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Distributed model predictive control of positive Markov jump systems

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Abstract

This paper proposes a new distributed model predictive control (DMPC) for positive Markov jump systems subject to uncertainties and constraints. The uncertainties refer to interval and polytopic types, and the constraints are described in the form of 1-norm inequalities. A linear DMPC framework containing a linear performance index, linear robust stability conditions, a stochastic linear co-positive Lyapunov function, a cone invariant set, and a linear programming based DMPC algorithm is introduced. A global positive Markov jump system is decomposed into several subsystems. These subsystems can exchange information with each other and each subsystem has its own controller. Using a matrix decomposition technique, the DMPC controller gain matrix is divided into nonnegative and non-positive components and thus the corresponding stochastic stability conditions are transformed into linear programming. By virtue of a stochastic linear co-positive Lyapunov function, the positivity and stochastic stability of the systems are achieved under the DMPC controller. A lower computation burden DMPC algorithm is presented for solving the min-max optimization problem of performance index. The proposed DMPC design approach is extended for general systems. Finally, an example is given to verify the effectiveness of the DMPC design.

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1. Introduction

Positive systems have drawn an increasing interest due to their interesting properties in theory and importance in practical applications [1–5]. This class of systems can model dynamic processes containing nonnegative quantities such as communication and traffic congestion [6], water systems [7], medical treatment [8], etc. Positive Markov jump systems (PMJSs) consist of positive subsystems and a Markov process. As a special class of positive systems, PMJSs are paid attention since they have powerful ability to model abrupt changes from operation environment, components, subsystems interconnections, and so on. Different from Markov jump systems (MJSs) without the positivity [9–13], PMJSs motivate new research approaches owing to the positivity requirement. In [14], it was shown that the mean square stability of PMJSs is equivalent to 1-moment stability and linear programming can be used to check the stability conditions. Some linear programming based necessary and sufficient conditions were derived for stochastic stability and ℓ_1 performance filter of PMJSs in [15]. In [16], a stochastic linear co-positive Lyapunov function was constructed and control synthesis of PMJSs was explored in terms of linear programming. Some mean square stability conditions were also presented in [17] for PMJSs with homogeneous transition probability by analyzing the time evolution of the first-order moment of the state. As we all know, the states of positive systems are kept in the nonnegative orthant. Based on the property, traditional Lyapunov stability theory with quadratic Lyapunov functions is replaced by the one with linear Lyapunov functions [18]. Accordingly, linear programming was used to check the corresponding conditions [19–21]. These properties of positive systems also bring some new research issues such as optimal control of PMJSs. Generally, the optimal control law of general systems was obtained by solving some Riccati equations [22] and Hamilton-Jacobi-Bellman equations [23]. However, these optimization approaches may not be valid for positive systems since the obtained optimal control cannot guarantee the positivity of positive systems. In addition, co-positive Lyapunov functions integrated with linear programming are more effective than quadratic Lyapunov functions integrated with linear matrix inequalities. Up to now, the optimal control of PMJSs is still an open issue.

Model predictive control (MPC) is extensively used to handle the constraints of systems [24–28]. MPC is a step-by-step optimization technique, in which an optimal control input is obtained at each time instant by solving an optimization problem. To deal with the optimal control of positive systems, a linear centralized MPC framework was established in [29–31]. As described in above positive systems literature, linear Lyapunov functions and linear programming are used in the linear MPC framework. It is also necessary to point out that the centralized MPC may be impractical and unsuitable for large-scale systems. PMJSs contain two classes of states: one is the continuous-time state $x(k)$ and the other one is the jump mode r_k. The MPC of PMJSs considers not only the performance of each mode but also the interconnection of subsystems. Practical positive systems such as communication networks [6], water systems [7], and medical treatment systems [8] are typical large-scale systems. These imply that the MPC may not be effective for PMJSs though there have been some MPC results on MJSs [32–35]. To overcome the drawbacks of MPC, distributed MPC (DMPC) is proposed and has received many concerns [36–38]. Under the DMPC framework, the plant mode is divided into several subsystems and then the controller of each subsystem is designed to reach a global performance. The collapse of the controller of some subsystem may not affect the stability of the systems since the controllers of other subsystems are still normal. DMPC reduces the computation burden of the MPC scheme of complex systems and increases the safety of systems. DMPC of stochastic systems has also been paid some attention. A DMPC method for the case that the states are not measurable was given in [39] by converting the probabilistic constraints into deterministic constraints. For the systems with parameter uncertainties, a stochastic DMPC algorithm based on generalized polynomial chaos expansions was developed in [40]. A DMPC design approach with Jacobi iterative algorithm was introduced for MJSs in [41]. Considering the systems with randomly occurring and Markov packet dropouts [42,43], an output feedback DMPC and a DMPC saturation control were proposed in terms of linear matrix inequalities, respectively.

By the above observation, it is clear that DMPC is powerful for dealing with the optimal control of complex systems and some significant achievements have been addressed in terms of linear matrix inequalities. Thus, two questions naturally arise: (i) whether the DMPC is available to PMJSs and (ii) how to establish a DMPC framework of PMJSs if the answer of (i) is positive. To the best of the authors' knowledge, there exist three challenges to solve the DMPC of PMJSs. First, the traditional DMPC may be unavailable. It has been shown in aforementioned literature that a linear approach is more tractable for positive systems. Most DMPC frameworks in literature are described in a quadratic form. Second, existing control approaches of PMJSs cannot be developed for the DMPC of PMJSs. How to guarantee the positivity of a system is one of difficult issues of positive systems. Under the DMPC framework, the underlying systems contain the input term of some subsystem and the input terms of other subsystems correlated to the subsystem. In this case, the positivity of the systems is more complex than general control synthesis of positive systems. Third, the DMPC algorithm involving linear matrix inequalities is less efficient for the DMPC of PMJSs. Computation burden has always been one obstacle of MPC applications in practice. Owing to the complexity of optimization algorithms involving linear matrix inequalities, it will lead to heavy computation burden and reduce the efficiency of MPC applications. The computation burden is still kept high though DMPC is introduced. These issues motivate us carry out the work.

This paper investigates the DMPC of PMJSs with interval and polytopic uncertainties, respectively. First, a linear performance index is introduced. Then, interval and polytopic uncertainties and linear constraints in the form of a 1-norm inequality are presented. Using a linear stochastic co-positive Lyapunov function, the DMPC controller of PMJSs is designed in terms of linear programming. A cone is constructed to guarantee the invariant property of the systems. Finally, a DMPC algorithm based on linear programming is provided. The contribution of the paper has three aspects: (i) a new DMPC framework is established for PMJSs, (ii) a linear programming based DMPC algorithm is presented, and (iii) the presented DMPC framework can be applied for MJSs and other issues of positive systems. The remainder of the paper is organized as follows. Section 2 describes the problem formation and gives some preliminaries of positive systems. Section 3 consists of four sections: The performance index, uncertainties, constraints, and a stochastic stability condition are presented in the first section; In the second section, the DMPC controller of PMJSs is designed; The third section handles the constraints; The last section explores the stochastic robust stability of PMJSs. In Section 4, the presented approach in Section 3 is developed for general systems. An example is provided in Section 5. Section 6 concludes the paper.

Notation: Let $\Re$, $\Re^n$, $\Re^n_+$, and $\Re^{n\times n}$ be the sets of real numbers, n-dimensional vectors, n-dimensional nonnegative vectors, and $n\times n$ matrices, respectively. Denote by $\mathbb{N}$ and $\mathbb{N}^+$ the sets of nonnegative and positive integers, respectively. For a vector $x=(x_1,\ldots,x_n)^T$, $x\succeq 0$ ($\succ 0$) and $x\preceq 0$ ($\prec 0$) mean that $x_i\geq 0$ ($x_i>0$) and $x_i\leq 0$ ($x_i<0$), $\forall i=1,\ldots,n$, respectively. For a matrix $A=[a_{ij}]\in\Re^{n\times n}$, the inequalities $A\succeq 0$ ($\succ 0$) and $A\preceq 0$ ($\prec 0$) mean that $a_{ij}\geq 0$

($a_{ij} > 0$) and $a_{ij} \leq 0$ ($a_{ij} < 0$), $\forall i, j = 1, \ldots, n$, respectively. The matrix I is the identical matrix with proper dimensions. The symbol Co refers to the convex hull. Let $e_n = (1, \ldots, 1)^T \in \Re^n$ and $e_n^{(i)} = (\underbrace{0, \ldots, 0}_{i-1}, 1, \underbrace{0, \ldots, 0}_{n-i})^T$. The symbol $\|x\|_1 = \sum_{i=1}^n |x_i|$ denotes the 1-norm of vector $x = (x_1, \ldots, x_n)^T$. $\mathbb{E}\{x\}$ stands for the expectation of stochastic variable x. Throughout the paper, the dimensions of vectors and matrices are assumed to be compatible if not stated.

2. Preliminaries

Consider a class of discrete-time time-varying stochastic systems:

$$x(k+1) = A(r_k)x(k) + B(r_k)u(k), \tag{1}$$

where $x(k) \in \Re^n$ and $u(k) \in \Re^m$ are the system state and the control input, respectively. The system matrices are unknown with $A(r_k) \in \Re^{n\times n}$ and $B(r_k) \in \Re^{n\times m}$. Let r_k be the mode and $\{r_k, k \in \mathbb{N}\}$ be a jumping process taking values in a finite set $\mathbb{S} = \{1, 2, \ldots, S\}, S \in \mathbb{N}^+$ with the transition rates: $P(r_{k+1} = q | r_k = p) = \pi_{pq}$, where $\pi_{pq} \geq 0$ and $\sum_{q=1}^S \pi_{pq} = 1, p, q \in \mathbb{S}$. For convenience, denote by A_i and B_i the system matrices for $r_k = i$.

In the following section, we introduce some preliminaries on positive systems and stochastic systems.

Definition 1 [1,2]**.** A system is positive if its state is nonnegative for any nonnegative initial state and any nonnegative control input.

Lemma 1 [1,2]**.** *A discrete-time system $x(k+1) = Ax(k) + Bu(k)$ is positive if and only if $A\succeq 0$ and $B\succeq 0$.*

Lemma 2. *The system (1) is positive if $A_i\succeq 0$ and $B_i\succeq 0$, $\forall i \in \mathbb{S}$.*

Lemma 2 is a direct extension of Lemma 1. Lemma 2 implies that a Markov jump system is positive if all its subsystems are positive.

Lemma 3 [1,2]**.** *For a matrix $A\succeq 0$, the following two conditions are equivalent:*

(i) *A is a Schur matrix;*
(ii) *There exists a vector $v \succ 0$ such that $(A - I)^T v \prec 0$.*

Give a positive system $x(k+1) = Ax(k)$. By Lemma 1, $A\succeq 0$. Choose a linear function $V(x(k)) = x(k)^T v$, where v is defined in Lemma 3. It is clear that $V(x(k))$ is positive definite since $x(k)\succeq 0$ and $v \succ 0$. Denote the difference of $V(x(k))$ by $\Delta V(x(k)) = V(x(k+1)) - V(x(k))$. By the term (ii) in Lemma 3, it is clear that $\Delta V(x(k)) < 0$, $\forall x(k) \neq 0$. Then, $V(x(k))$ is a Lyapunov function of the considered positive system. Such a linear function is called linear co-positive Lyapunov function and will be used later to reach the stability of the systems considered in the paper.

Definition 2 [3]**.** The positive system (1) with $u(k) = 0$ is mean-square stable if for given initial state $x(k_0)$ and initial mode r_0, $\mathbb{E}\{\|x(k)\|_1 : x(k_0), r_0\} \to 0$ as $k \to \infty$.

3. DMPC of PMJSs

This section is divided into four sections. In the first section, the global system is decomposed into several subsystems. A linear performance index is constructed and uncertainties and constraints are introduced. The second section proposes the DMPC controller design. In the third section, the constraints are handled. The last section addresses the stochastic stability of PMJSs.

3.1. Linear DMPC framework

Two classes of uncertainties are considered for system (1). The first class is interval uncertainty:

$$\Omega_1(r_k) := \{[A(r_k)\ B(r_k)] \big| \underline{A}(r_k) \preceq A(r_k) \preceq \overline{A}(r_k), \underline{B}(r_k) \preceq B(r_k) \preceq \overline{B}(r_k)\}, \tag{2}$$

where $\underline{A}(r_k) \in \Re^{n\times n}$, $\underline{B}(r_k) \in \Re^{n\times m}$ and $\overline{A}(r_k) \in \Re^{n\times n}$, $\overline{B}(r_k) \in \Re^{n\times m}$ are the lower and upper bound matrices, respectively, and satisfy that $\underline{A}(r_k) \succeq 0$, $\underline{B}(r_k) \succeq 0$. The second one is polytopic uncertainty:

$$\Omega_2(r_k) := Co\{[A^{(1)}(r_k)\ B^{(1)}(r_k)], \ldots, [A^{(L)}(r_k)\ B^{(L)}(r_k)]\}, \tag{3}$$

where $A^{(\ell)}(r_k) \in \Re^{n\times n}$, $B^{(\ell)}(r_k) \in \Re^{n\times m}$, $\ell = 1, 2, \ldots, L$ are the vertex matrices and satisfy that $A^{(\ell)}(r_k) \succeq 0, B^{(\ell)}(r_k) \succeq 0$.

The model (1) can be decomposed into N subsystems:

$$\begin{pmatrix} x_{11}(k+1) \\ \vdots \\ x_{ii}(k+1) \\ \vdots \\ x_{NN}(k+1) \end{pmatrix} = A_i(r(k)) \begin{pmatrix} x_{11}(k) \\ \vdots \\ x_{ii}(k) \\ \vdots \\ x_{NN}(k) \end{pmatrix} + (B_1(r(k)), \ldots, B_i(r(k)), \ldots, B_N(r(k))) \begin{pmatrix} u_1(k) \\ \vdots \\ u_i(k) \\ \vdots \\ u_N(k) \end{pmatrix} \tag{4}$$

where $x_{ii}(k) \in \Re^{n_{ii}}$ and $u_i(k) \in \Re^{m_i}$. Then, the distributed systems with PMJSs can be given as:

$$x_i(k+1) = A_i(r(k))x_i(k) + B_i(r(k))u_i(k) + \sum_{j=1, j\neq i}^{N} B_j(r(k))u_j(k), \tag{5}$$

where $x_i(k) \in \Re^n$ and $u_i(k) \in \Re^{m_i}$ are the state and control input of the ith subsystem, respectively, and $m = \sum_{i=1}^{N} m_i$. It should be pointed out that the state $x_i(k) = (x_{11}, \ldots, x_{ii}, \ldots, x_{NN})^T$ contains all states of the system (4), $A_i(r(k)) = A(r(k))$, and $B_i(r(k))$ is the ith column of $B(r(k))$. By (2) and (3), the uncertainties of the system matrices in (5) are rewritten as:

$$\Omega_1(r_k) = \{[A_i(r_k)\ B_i(r_k)] \big| \underline{A}_i(r_k) \preceq A_i(r_k) \preceq \overline{A}_i(r_k), \underline{B}_i(r_k) \preceq B_i(r_k) \preceq \overline{B}_i(r_k)\}, \tag{6}$$

and

$$\Omega_2(r_k) = Co\{[A_i^{(1)}(r_k)\ B_i^{(1)}(r_k)], \ldots, [A_i^{(L)}(r_k)\ B_i^{(L)}(r_k)]\}. \tag{7}$$

In the MPC and DMPC literature [32–34,41–43], polytopic uncertain has been extensively employed to describe the uncertainty of systems owing to its powerful in modeling time-varying and nonlinear processes. In this paper, we first follow the polytopic uncertainty used in literature. On the other hand, we also introduce interval uncertainty to PMJSs. Interval uncertainty can model a large class of uncertain systems by giving the lower and upper bounds of system matrices. Give a system $x(k+1) = Ax(k)$, where $\underline{A} \preceq A \preceq \overline{A}$. Suppose that $\underline{A} \succeq 0$ and $\overline{A}$ is a Shur matrix. First, we have $A \succeq \underline{A} \succeq 0$, which implies that the considered system is positive. By Lemma 3, there exists a vector $v \succ 0$ such that $(\overline{A} - I)^T v \prec 0$. Thus, $(A - I)^T v \prec (\overline{A} - I)^T v \prec 0$. This reveals that the considered system is stable. In summary, the positivity and stability of an interval positive system can be reached by guaranteeing the positivity of the lower bound of the system and the stability of the upper bound of the system, respectively. This is a good property of interval positive systems whereas it is not for general interval systems (non-positive). Some statements about interval uncertainty can refer to [30]. The challenge to tackle the interval uncertainty lies in how to design a controller for guaranteeing either the positivity of the lower bound of interval uncertain systems or the stability of the upper bound of interval uncertain systems.

For general systems, the constraint conditions are usually presented based on the Euclidean norm [25,26]. Note the fact that the states of positive systems are nonnegative. Thus, the following constraint conditions are introduced for the system (5):

$$||x_i(k)||_1 \le \delta, \tag{8a}$$

$$||u_i(k)||_1 \le \eta, \tag{8b}$$

where $\delta > 0$ and $\eta > 0$ are given constants. Some similar constraint conditions have also been used in [29] and [30].

The objective of this paper is to design a set of DMPC controllers:

$$\begin{aligned} u_i(k+s|k) &= F_{ii}(k, r_{k+s|k}) x_{ii}(k+s|k) + \sum_{j=1, j\neq i}^{N} F_{ij}(k, r_{k+s|k}) x_{jj}(k+s|k), \\ &= F_i(k, r_{k+s|k}) x_i(k+s|k), \quad i = 1, 2, \ldots, N, \quad s = 1, 2, \ldots, \infty \end{aligned} \tag{9}$$

such that the system (5) is positive and stochastic stability by solving the optimization:

$$\min_{\substack{u_i(k+s|k) \\ i=1,2,\ldots,N, s\ge 0}} \max_{\substack{[A_i(r_k)\ \ B_i(r_k)] \in \Omega_1 (\text{or}, \Omega_2) \\ i=1,2,\ldots,N, r_{k+s|k}\in \mathbb{S}}} J_i(k) \quad \text{subject to} \quad (5) \text{ and } (8), \tag{10}$$

with the performance index function:

$$\begin{aligned} J_i(k) = \mathbb{E}_k \Big\{ &\sum_{s=0}^{\infty} \big(x_i^T(k+s|k) \varsigma(k+s|k) + u_i^T(k+s|k) \varrho_i(k+s|k) \\ &+ \sum_{j=1, j\neq i}^{N} u_j^{*T}(k+s|k) \varrho_j(k+s|k) \big) \Big\}, \end{aligned} \tag{11}$$

where $x_i(k+s|k)$ and $u_i(k+s|k)$ are the state and input predicted at time instant k, $\varsigma(k+s|k) \succ 0$, $\varrho_i(k+s|k) \prec 0$, $\varrho_j(k+s|k) \prec 0$, and $u_j^*(\cdot)$ is the solution obtained from a previous iteration and kept fixed in the current iteration. As stated in [30], there does not exist any nonnegative control law such that a discrete-time positive system is stable. Based on this point, we assume that the DMPC control law to be designed is negative, that is, $u_i^T(k+s|k) \prec 0$. Consequently, the corresponding parameters $\varsigma(k+s|k), \varrho_i(k+s|k)$, and $\varrho_j(k+s|k)$ are introduced to guarantee the validity of the performance index function.

3.2. DMPC design

First, a stochastic linear co-positive Lyapunov function is constructed for the system (5). Then, a stochastic stability condition is derived. Finally, a DMPC controller design is proposed for the system (5).

By Eqs. (5) and (9), the closed-loop system is:

$$\begin{aligned} x_i(k+s+1|k) &= \Big(A_i(p)+B_i(p)F_i(k,p)\Big)x_i(k+s|k) + \sum_{j=1,j\neq i}^{N} B_j(p)F_j^*(k,p)x_j(k+s|k) \\ &= \Big(A_i^*(p)+B_i(p)F_i(k,p)\Big)x_i(k+s|k), \end{aligned} \tag{12}$$

where $r_{k+s|k}=p$, $A_i^*(p)=A_i(p)+\sum_{j=1,j\neq i}^{N} B_j(p)F_j^*(k,p)$ and the second equation follows from $x_i(k+s|k)=x_j(k+s|k)$. Construct a stochastic linear co-positive Lyapunov function:

$$V_i(k+s|k) = x_i^T(k+s|k)v_i(k,p), \tag{13}$$

where $v_i(k,p)\succ 0$, $v_i(k,p)\in\Re^n$. To obtain the bound of the performance index in Eq. (10), a robust stability condition is introduced:

$$\begin{aligned} V_i(k+s+1|k) - V_i(k+s|k) \leq -\Big(& x_i^T(k+s|k)\varsigma(k+s|k) + u_i^T(k+s|k)\varrho_i(k+s|k) \\ &+ \sum_{j=1,j\neq i}^{N} u_j^{*T}(k+s|k)\varrho_j(k+s|k)\Big). \end{aligned} \tag{14}$$

Theorem 1. *(Controller design)* ***(a) Interval uncertainty).*** *If there exist constants $\hbar>1$, $\gamma_i(k)>0$ and $\Re^n$ vectors $v_i(k,p)\succ 0, \xi_i^{(\iota)}(k,p)\prec 0, \overline{\xi}_i(k,p)\prec 0$ such that*

$$A_i^{*T}(p)\sum_{q=1}^{S}\pi_{pq}v_i(k,q) + \overline{\xi}_i(k,p) - v_i(k,p) + \varsigma^*(p) \prec 0, \tag{15a}$$

$$\begin{aligned} &\underline{A}_i(p)e_{m_i}^T\left(\underline{B}_i^T(p)\sum_{q=1}^{S}\pi_{pq}v_i(k,q)+\varrho_i(p)\right) + \hbar\sum_{j=1,j\neq i}^{N}\overline{B}_j(p)\sum_{\iota=1}^{m_i}e_{m_i}^{(\iota)}\xi_j^{(\iota)T}(k,p) \\ &+\overline{B}_i(p)\sum_{\iota=1}^{m_i}e_{m_i}^{(\iota)}\xi_i^{(\iota)T}(k,p)\succeq 0, \end{aligned} \tag{15b}$$

$$e_{m_i}^T\left(\underline{B}_i^T(p)\sum_{q=1}^{S}\pi_{pq}v_i(k,q)+\varrho_i(p)\right) \leq \hbar e_{m_j}^T\left(\underline{B}_j^T(p)\sum_{q=1}^{S}\pi_{pq}v_j(k,q)+\varrho_j(p)\right), \tag{15c}$$

$$\underline{B}_i^T(p)\sum_{q=1}^{S}\pi_{pq}v_i(k,q)+\varrho_i(p)\succ 0, \tag{15d}$$

$$\xi_i^{(\iota)}(k,p)\preceq\overline{\xi}_i(k,p),\quad \iota=1,2,\ldots,m_i, \tag{15e}$$

and

$$x_i^T(k|k)v_i(k,p) \le \gamma_i(k) \tag{16}$$

hold $\forall p \in \mathbb{S}, \forall (i,j) \in \{1,\dots,N\} \times \{1,\dots,N\}, i \neq j,$ *then under the control law*

$$u_i(k+s|k) = F_i(k,p)x_i(k+s|k) = \frac{\sum_{\iota=1}^{m_i} e_{m_i}^{(\iota)} \xi_i^{(\iota)T}(k,p)}{e_{m_i}^T(\underline{B}_i^T(p)\sum_{q=1}^S \pi_{pq} v_i(k,q) + \varrho_i(p))} x_i(k+s|k), \tag{17}$$

the interval uncertain system (5) *is positive and satisfies the condition* (14), *where* $\overline{A}_i^*(p) = \overline{A}_i(p) + \sum_{j=1, j\neq i}^N \underline{B}_j(p)F_j^*(k,p)$.

(b) Polytopic uncertainty. *If there exist constants* $\hbar > 1$, $\gamma_i(k) > 0$ *and vectors* $v_i(k,p) \succ 0$ *with* $v_i(k,p) \in \Re^n$, $\rho_i(p) \succ 0$ *with* $\rho_i(p) \in \Re^{m_i}$, $\xi_i^{(\iota)}(k,p) \in \Re^n$, $\overline{\xi}_i(k,p) \prec 0$ *with* $\overline{\xi}_i(k,p) \in \Re^n$ *such that*

$$A_i^{(l)*T}(p)\sum_{q=1}^S \pi_{pq} v_i(k,q) + \overline{\xi}_i(k,p) - v_i(k,p) + \varsigma^*(p) \prec 0, \tag{18a}$$

$$A_i^{(l)}(p)e_{m_i}^T \rho_i(p) + \hbar \sum_{j=1, j\neq i}^N B_j^{(l)}(p)\sum_{\iota=1}^{m_j} e_{m_j}^{(\iota)} \xi_j^{(\iota)T}(k) + B_i^{(l)}(p)\sum_{\iota=1}^{m_i} e_{m_i}^{(\iota)} \xi_i^{(\iota)T}(k) \succeq 0, \tag{18b}$$

$$B_i^{(l)T}(p)\sum_{q=1}^S \pi_{pq} v_i(k,q) + \varrho_i \succeq \rho_i(p), \tag{18c}$$

$$e_{m_i}^T \rho_i(p) \le \hbar e_{m_j}^T \rho_j(p), \tag{18d}$$

$$\xi_i^{(\iota)}(k,p) \preceq \overline{\xi}_i(k,p), \ \ \iota = 1,2,\dots,m_i, \tag{18e}$$

and (16) *hold* $\forall l \in \{1,\dots,L\}, \forall p \in \mathbb{S}, \forall (i,j) \in \{1,\dots,N\} \times \{1,\dots,N\}, i \neq j,$ *then under the control law*

$$u_i(k+s|k) = F_i(k,p)x(k+s|k) = \frac{\sum_{\iota=1}^{m} e_{m_i}^{(\iota)} \xi_i^{(\iota)T}(k,p)}{e_{m_i}^T \rho_i(p)} x(k+s|k), \tag{19}$$

the polytopic system (5) *is positive and satisfies the condition* (14), *where* $A_i^{(l)*}(p) = A_i^{(l)}(p) + \sum_{j=1, j\neq i}^{N} B_j^{(l)}(p)F_j^*(k,p)$.

Proof. Along the system (12), the difference of the Lyapunov function in Eq. (13) is

$$\begin{aligned}
\mathbb{E}_{k+s|k}(\Delta V_i(k+s|k)) &= \mathbb{E}_{k+s|k}\big(V_i(k+s+1|k) - V_i(k+s|k)\big) \\
&= \mathbb{E}_{k+s|k}\big(x_i^T(k+s+1|k)v_i(k, r_{k+s+1|k})\big) - x_i^T(k+s|k)v_i(k,p) \\
&= x_i^T(k+s|k)\Big(\big(A_i^*(p) + B_i(p)F_i(k,p)\big)^T \mathbb{E}_{k+s|k}(v_i(k, r_{k+s+1|k})) - v_i(k,p)\Big).
\end{aligned} \tag{20}$$

Combining Eqs. (9), (14), and (20) gives

$$x_i^T(k+s|k)\Big(\big(A_i^*(p) + B_i(p)F_i(k,p)\big)^T \mathbb{E}_{k+s|k}(v_i(k, r_{k+s+1|k})) - v_i(k,p)$$

$$
+\varsigma(p)+F_i^T(k,p)\varrho_i(p)+\sum_{j=1,j\neq i}^{N}F_j^{*T}(k,p)\varrho_j(p)\Big)
\leq 0. \tag{21}
$$

By the expectation property of the Markov process, it follows that $\mathbb{E}_{k+s|k}(v_i(k, r_{k+s+1|k})) = \sum_{q=1}^{S}\pi_{pq}v_i(k,q)$. Then, the inequality (21) is equivalent to

$$
x_i^T(k+s|k)\Big(\big(A_i^*(p)+B_i(p)F_i(k,p)\big)^T\sum_{q=1}^{S}\pi_{pq}v_i(k,q)-v_i(k,p)+\varsigma^*(p)+F_i^T(k,p)\varrho_i(p)\Big)\leq 0, \tag{22}
$$

where $\varsigma^*(p)=\varsigma(p)+\sum_{j=1,j\neq i}^{N}F_j^{*T}(k,p)\varrho_j(p)$.

(a) Interval uncertainty. First, the positivity of the interval uncertain system (5) is discussed. By (15d), it follows that $e_{m_i}^T(\underline{B}_i^T(p)\sum_{q=1}^{S}\pi_{pq}v_i(k,q)+\varrho_i(p))>0$. By (15b),

$$
\underline{A}_i(p)+\hbar\sum_{j=1,j\neq i}^{N}\overline{B}_j(p)\frac{\sum_{\iota=1}^{m_i}e_{m_i}^{(\iota)}\xi_j^{(\iota)T}(k,p)}{e_{m_i}^T(\underline{B}_i^T(p)\sum_{q=1}^{S}\pi_{pq}v_i(k,q)+\varrho_i(p))}
+\overline{B}_i(p)\frac{\sum_{\iota=1}^{m_i}e_{m_i}^{(\iota)}\xi_i^{(\iota)T}(k,p)}{e_{m_i}^T(\underline{B}_i^T(p)\sum_{q=1}^{S}\pi_{pq}v_i(k,q)+\varrho_i(p))}\succeq 0.
$$

It is easy to obtain from (15c) that $\frac{\hbar}{e_{m_i}^T(\underline{B}_i^T(p)\sum_{q=1}^{S}\pi_{pq}v_i(k,q)+\varrho_i(p))}\geq\frac{1}{e_{m_j}^T(\underline{B}_j^T(p)\sum_{q=1}^{S}\pi_{pq}v_j(k,q)+\varrho_j(p))}$. Together with $\xi_j^{(\iota)}(k,p)\prec 0$, (6), and (17) gives

$$
\begin{aligned}
0\preceq\;&\underline{A}_i(p)+\hbar\sum_{j=1,j\neq i}^{N}\overline{B}_j(p)\frac{\sum_{\iota=1}^{m_i}e_{m_i}^{(\iota)}\xi_j^{(\iota)T}(k,p)}{e_{m_i}^T(\underline{B}_i^T(p)\sum_{q=1}^{S}\pi_{pq}v_i(k,q)+\varrho_i(p))}\\
&+\overline{B}_i(p)\frac{\sum_{\iota=1}^{m_i}e_{m_i}^{(\iota)}\xi_i^{(\iota)T}(k,p)}{e_{m_i}^T(\underline{B}_i^T(p)\sum_{q=1}^{S}\pi_{pq}v_i(k,q)+\varrho_i(p))}\\
\preceq\;&\underline{A}_i(p)+\sum_{j=1,j\neq i}^{N}\overline{B}_j(p)\frac{\sum_{\iota=1}^{m_j}e_{m_j}^{(\iota)}\xi_j^{(\iota)T}(k,p)}{e_{m_j}^T(\underline{B}_j^T(p)\sum_{q=1}^{S}\pi_{pq}v_j(k,q)+\varrho_j(p))}\\
&+\overline{B}_i(p)\frac{\sum_{\iota=1}^{m_i}e_{m_i}^{(\iota)}\xi_i^{(\iota)T}(k,p)}{e_{m_i}^T(\underline{B}_i^T(p)\sum_{q=1}^{S}\pi_{pq}v_i(k,q)+\varrho_i(p))}\\
=\;&\underline{A}_i(p)+\sum_{j=1,j\neq i}^{N}\overline{B}_j(p)F_j(k,p)+\overline{B}_i(p)F_i(k,p)\\
\preceq\;&A_i(p)+\sum_{j=1,j\neq i}^{N}B_j(p)F_j(k,p)+B_i(p)F_i(k,p),
\end{aligned}
$$

which implies that the pth mode of the interval uncertain system (5) is positive by Lemma 1. Thus, the interval uncertain system (5) is positive by Lemma 2.

Next, consider the validity of the condition (14). By (15c) and (17), the following inequalities hold:

$$
\begin{aligned}
&F_i^T(k,p)B_i^T(p)\sum_{q=1}^{S}\pi_{pq}v_i(k,q)+F_i^T(k,p)\varrho_i(p)\\
&\preceq \frac{\sum_{\iota=1}^{m_i}\xi_i^{(\iota)}(k,p)e_{m_i}^{(\iota)T}}{e_{m_i}^T(\underline{B}_i^T(p)\sum_{q=1}^{S}\pi_{pq}v_i(k,q)+\varrho_i(p))}(B_i^T(p)\sum_{q=1}^{S}\pi_{pq}v_i(k,q)+\varrho_i(p))\\
&\preceq \frac{\sum_{\iota=1}^{m_i}\overline{\xi}_i(k,p)e_{m_i}^{(\iota)T}}{e_{m_i}^T(\underline{B}_i^T(p)\sum_{q=1}^{S}\pi_{pq}v_i(k,q)+\varrho_i(p))}(B_i^T(p)\sum_{q=1}^{S}\pi_{pq}v_i(k,q)+\varrho_i(p))\\
&=\frac{\overline{\xi}_i(k,p)e_{m_i}^T(B_i^T(p)\sum_{q=1}^{S}\pi_{pq}v_i(k,q)+\varrho_i(p))}{e_{m_i}^T(\underline{B}_i^T(p)\sum_{q=1}^{S}\pi_{pq}v_i(k,q)+\varrho_i(p))}\\
&\preceq \overline{\xi}_i(k,p).
\end{aligned}
\tag{23}
$$

From Eqs. (6) and (23),

$$
\begin{aligned}
&\big(A_i^*(p)+B_i(p)F_i(k,p)\big)^T\sum_{q=1}^{S}\pi_{pq}v_i(k,q)-v_i(k,p)+\varsigma^*(p)+F_i^T(k,p)\varrho_i(p)\\
&\preceq A_i^{*T}(p)\sum_{q=1}^{S}\pi_{pq}v_i(k,q)+\overline{\xi}_i(k,p)-v_i(k,p)+\varsigma^*(p).
\end{aligned}
\tag{24}
$$

Combining the fact $x_i^T(k+s|k)\succeq 0$, (15a), and (24) concludes that the condition (22) holds, that is, the condition (14) is satisfied. Noting the facts $e_{m_i}^T(\underline{B}_i^T(p)\sum_{q=1}^{S}\pi_{pq}v_i(k,q)+\varrho_i(p))>0$ and $\sum_{\iota=1}^{m_i}e_{m_i}^{(\iota)}\xi_i^{(\iota)T}(k,p)\prec 0$, it follows that $F_i(k,p)\prec 0$. Thus, $u_i(k+s|k)\prec 0$.

Finally, the upper bound of the performance index in (10) is obtained. Taking the expectation for both sides of the condition (14) and summing it up from $s=0$ to ∞ give

$$
\begin{aligned}
\mathbb{E}_k(V_i(\infty|k))-\mathbb{E}_k(V_i(k)) \quad &\le -\mathbb{E}_k\Big(\sum_{s=0}^{\infty}\Big(x_i^T(k+s|k)\varsigma(k+s|k)+u_i^T(k+s|k)\varrho_i(k+s|k)\\
&\qquad+\sum_{j=1,j\neq i}^{N}u_j^{*T}(k+s|k)\varrho_j(k+s|k)\Big)\Big).
\end{aligned}
$$

From (14), it is easy to have $\mathbb{E}_k(V_i(\infty|k))=0$. Thus, $J_i(k)\le\mathbb{E}_k(V_i(k))=x_i^T(k|k)v_i(k,p)$. Let $\gamma_i(k)$ be the upper bound of $J_i(k)$ satisfying $\gamma_i(k)\ge x_i^T(k|k)v_i(k,p)$, which is just the condition (16).

(b) Polytopic uncertainty. Using (18b) follows that

$$
A_i^{(l)}(p)+\hbar\sum_{j=1,j\neq i}^{N}B_j^{(l)}(p)\frac{\sum_{\iota=1}^{m_j}e_{m_j}^{(\iota)}\xi_j^{(\iota)T}(k)}{e_{m_i}^T\rho_i(p)}+B_i^{(l)}(p)\frac{\sum_{\iota=1}^{m_i}e_{m_i}^{(\iota)}\xi_i^{(\iota)T}(k)}{e_{m_i}^T\rho_i(p)}\succeq 0.
$$

By (18d), it derives that $\frac{\hbar}{e_{m_i}^T\rho_i(p)}\ge\frac{1}{e_{m_j}^T\rho_j(p)}$. Together with $\xi_j^{(\iota)}(k,p)\prec 0$, (7), and (19) gives

$$
0\preceq A_i^{(l)}(p)+\hbar\sum_{j=1,j\neq i}^{N}B_j^{(l)}(p)\frac{\sum_{\iota=1}^{m_j}e_{m_j}^{(\iota)}\xi_j^{(\iota)T}(k)}{e_{m_i}^T\rho_i(p)}+B_i^{(l)}(p)\frac{\sum_{\iota=1}^{m_i}e_{m_i}^{(\iota)}\xi_i^{(\iota)T}(k)}{e_{m_i}^T\rho_i(p)}
$$

$$\preceq A_i^{(l)}(p) + \sum_{j=1, j\neq i}^{N} B_j^{(l)}(p) \frac{\sum_{\iota=1}^{m_j} e_{m_j}^{(\iota)} \xi_j^{(\iota)T}(k)}{e_{m_j}^T \rho_j(p)} + B_i^{(l)}(p) \frac{\sum_{\iota=1}^{m_i} e_{m_i}^{(\iota)} \xi_i^{(\iota)T}(k)}{e_{m_i}^T \rho_i(p)}$$

$$= A_i^{(l)}(p) + \sum_{j=1, j\neq i}^{N} B_j^{(l)}(p) F_j(k,p) + B_i^{(l)}(p) F_i(k,p). \tag{25}$$

From Eqs. (7) and (12), the polytopic uncertain system (5) can be rewritten as:

$$x_i(k+s+1|k) = \sum_{l=1}^{L} \lambda_l \Big(A_i^{(l)}(p) + \sum_{j=1, j\neq i}^{N} B_j^{(l)}(p) F_j(k,p) + B_i^{(l)}(p) F_i(k,p) \Big) x_i(k+s|k), \tag{26}$$

where $\sum_{l=1}^{L} \lambda_l = 1, \lambda_l \geq 0$. From (25), the system (26) is positive by Lemma 1.

By (18c) and (19), it holds that

$$F_i^T(k,p) B_i^{(l)T}(p) \sum_{q=1}^{S} \pi_{pq} v_i(k,q) + F_i^T(k,p) \varrho_i(p) \tag{27}$$

$$\preceq \frac{\sum_{\iota=1}^{m_i} \overline{\xi}_i(k,p) e_{m_i}^{(\iota)T}}{e_{m_i}^T \rho_i(p)} \big(B_i^{(l)T}(p) \sum_{q=1}^{S} \pi_{pq} v_i(k,q) + \varrho_i(p) \big)$$

$$= \frac{\overline{\xi}_i(k,p) e_{m_i}^T \big(B_i^{(l)T}(p) \sum_{q=1}^{S} \pi_{pq} v_i(k,q) + \varrho_i(p) \big)}{e_{m_i}^T \rho_i(p)}$$

$$\preceq \overline{\xi}_i(k,p). \tag{27}$$

From (7) and (27),

$$\big(A_i^*(p) + B_i(p) F_i(k,p) \big)^T \sum_{q=1}^{S} \pi_{pq} v_i(k,q) - v_i(k,p) + \varsigma^*(p) + F_i^T(k,p) \varrho_i(p)$$

$$= \sum_{l=1}^{L} \lambda_l \Big(\big(A_i^{(l)*}(p) + B_i^{(l)}(p) F_i(k,p) \big)^T \sum_{q=1}^{S} \pi_{pq} v_i(k,q) - v_i(k,p) + \varsigma^*(p) + F_i^T(k,p) \varrho_i(p) \Big)$$

$$\preceq \sum_{l=1}^{L} \lambda_l \Big(A_i^{(l)*T}(p) \sum_{q=1}^{S} \pi_{pq} v_i(k,q) + \overline{\xi}_i(k,p) - v_i(k,p) + \varsigma^*(p) \Big). \tag{28}$$

By Eqs. (18a) and (28), it can be derived that the condition (22) holds, that is, the condition (14) is satisfied.

Finally, the upper bound of the performance index in Eq. (10) can be achieved by Eq. (16) and the corresponding proof is the same as that in the interval uncertain case. □

Remark 1. In (17), a matrix decomposition technique is employed for the design of DMPC controller gain matrix $F_i(k,p)$. A decomposed form of $F_i(k,p)$ in (17) is given by:

$$F_i(k,p) = \frac{1}{e_{m_i}^T (\underline{B}_i^T(p) \widehat{v}_i(k) + \varrho_i(p))} \Big(e_{m_i}^{(1)} \xi_i^{(1)T}(k,p) + e_{m_i}^{(2)} \xi_i^{(2)T}(k,p) + \cdots + e_{m_i}^{(m_i)} \xi_i^{(m_i)T}(k,p) \Big)$$

$$= \frac{1}{e_{m_i}^T (\underline{B}_i^T(p) \widehat{v}_i(k) + \varrho_i(p))} \Big(\big(\underbrace{\xi_i^{(1)}(k,p), \mathbf{0}, \ldots, \mathbf{0}}_{m_i} \big)^T + \big(\underbrace{\mathbf{0}, \xi_i^{(2)}(k,p), \mathbf{0}, \ldots, \mathbf{0}}_{m_i} \big)^T$$

$$+\big(\underbrace{\mathbf{0},\ldots,\mathbf{0},\xi_i^{(m_i)}(k,p)}_{m_i}\big)^T\Big),$$

where $\mathbf{0}=(0,\ldots,0)^T\in\Re^n$. Under the decomposed form, it is easy to transform the robust stable condition (14) into (15a). Moreover, the positivity condition (15b) is obtained. It is clear that Eq. (15a) and (15b) are solved in terms of linear programming. A similar method is used in Eq. (19).

Remark 2. In [30], it was shown that there does not exist a nonnegative feedback controller such that a discrete-time positive system is stable. Hence, the MPC controller in [30] was required to be negative. This paper follows the method in [30]. It should be pointed out that there may exist a controller with nonnegative and non-positive components such that the corresponding system is stable. Thus, it would be interesting to remove the sign restriction of the DMPC in Eqs. (17) and (19) in future work.

Remark 3. For general systems (non-positive), a Lyapunov function is usually constructed in a quadratic form: $V(x(k))=x^T(k)Px(k)$, where P is a positive definite matrix with compatible dimension. Such a quadratic form can guarantee the positive definite property of the Lyapunov function. Stochastic Lyapunov functions with a quadratic form are widely used for MJSs [9–13]. The state of positive systems is nonnegative. Therefore, a linear function: $V(x(k))=x^T(k)v$ can be chosen as the Lyapunov function of positive systems, where $v\succ 0$ with compatible dimension. Under the linear Lyapunov function, linear programming is naturally employed as computation tool. A linear approach including linear Lyapunov functions and linear programming has been employed for positive systems [4–8]. Consequently, linear stochastic Lyapunov functions and linear programming have also been developed for PMJSs [15–21]. In [39–43], the DMPC has been considered. In [41–43], linear matrix inequalities were chosen as the computation method. For PMJSs, the number of linear matrix inequalities conditions may increase twice since the positivity of the systems are required besides the stability. Consequently, the traditional DMPC framework is not very suitable for PMJSs. From the Introduction, it is not hard to find that a linear approach is more effective for positive systems than other approaches. Therefore, Theorem 1 proposes a linear Lyapunov function associated with linear programming approach for the DMPC controller design of PMJSs.

Remark 4. To guarantee the linearity of the conditions (15) and (18), the parameter $\hbar$ is given. Two questions yield: (i) how to choose the parameter, and (ii) whether the parameter will bring conservatism to Theorem 1. For the first question, a suggested algorithm is provided later. A discussion on the second question is given as follows. For two positive real numbers a and b, a fact is that there must exist a constant $\hbar>1$ such that $a<\hbar b$. This reveals that the conditions (15d) and (18d) do not increase the restriction for the conditions (15) and (18), respectively. Take (15b) and (18b) into account. If the inequality $A+BK\succeq 0$ holds, then there must exist a constant $\hbar>1$ such that $A+\hbar BK'\succeq 0$ holds, where $A\succeq 0$, $B\succeq 0$, and $K=\hbar K'$. Based on this point, the conditions (15b) and (18b) do not bring the conservatism to the conditions (15) and (18), respectively.

To obtain the value of $\gamma_i(k)$, a linear programming can be implemented:

$$\min_{v_i(k,p),\xi_i^{(\iota)}(k,p),\bar{\xi}_i(k,p)} \gamma_i(k) \quad \text{subject to (15) and (16) (or, (18) and (16))}. \tag{29}$$

To choose the value of $\hbar$ and compute (29), a suggestive algorithm is introduced as follows:

Algorithm 1

Step 1: Let $\hbar \in [1, \bar{\hbar}]$, where $\bar{\hbar}$ is a given constant. Set $\hbar_i = 1 + \sigma_i(\bar{\hbar} - 1)$, where $i = 1, 2, \ldots, M, M \in \mathbb{N}$ and σ_i is a random number in the interval [0,1].

Step 2: Substitute $\hbar_i, i = 1, 2, \ldots, M$, into (15) (or, (18)). If the condition (15) (or, (18)) is feasible, denote the corresponding values of $\hbar_i$ as $\hbar_{fi}, i = 1, 2, \ldots, M_f$, where $M_f \leq M$ and go to Step 4. Otherwise, go to Step 3.

Step 3: Choose $\hbar \in [\bar{\hbar}, \bar{\hbar}']$, where $\bar{\hbar}'$ is a given constant. Set $\hbar_i = 1 + \sigma_i(\bar{\hbar}' - \bar{\hbar})$, where $i = 1, 2, \ldots, M', M' \in \mathbb{N}$ and σ_i is a random number in the interval [0,1]. Repeat Step 2.

Step 4: Implement the optimization (29) for each $\hbar_{fi}, i = 1, 2, \ldots, M_f$ and find the minimal value of $\gamma_i(k)$.

Theorem 2. *If the optimization (29) is feasible at time instant k for the initial state $x(k)$ and initial mode r_k, then the optimization (29) is also feasible at any time instant $k' \geq k$. Moreover, the DMPC controller obtained from (29) guarantees the stability of the system (5) with interval/polytopic uncertainties in the mean-square sense.*

Proof. (a) Interval uncertainty. Assume that the optimization (29) is feasible at the sample time instant k. Denote the optimization solution as $\Phi_i(k) = \{\gamma_i^*(k), F_i^*(k, p), v_i(k, p), \xi_i^{(\iota)}(k, p), \overline{\xi}_i(k, p), \hbar\}$ and the control sequence as $\mathcal{U}_i(k) = \{u_i(k), u_i(k+1|k), \ldots, u_i(k+M-1|k)\}$, where M is the predictive step. By (16), $V_i(k|k) \leq \gamma_i^*(k)$. At the $(k+1)$th sample time instant, construct a feasible solution as $\Phi_i(k+1) = \{\gamma_i(k+1) = \gamma_i^*(k), F_i(k+1, p) = F_i^*(k, p), v_i(k, p), \xi_i^{(\iota)}(k, p), \overline{\xi}_i(k, p), \hbar\}$ and a control sequence as $\mathcal{U}_i(k+1) = \{u_i(k+1|k), \ldots, u_i(k+M-1|k), 0\}$. First, $\Phi_i(k+1)$ is a solution to (15). Then, the condition (14) holds, that is, $V(k+1|k+1) \leq V(k|k)$. Thus, $V(k+1|k+1) \leq \gamma_i^*(k) = \gamma_i(k+1)$. This implies that the optimization (29) is feasible at the sample time instant $k+1$. By a recursive induction, the feasibility of the optimization (29) is reached.

From (15) and (20), we have $\mathbb{E}_{k+s|k}\big(V_i(k+s+1|k)\big) \leq V_i(k+s|k), \forall s = 1, 2, \ldots$. That is to say, $\mathbb{E}_{k+s|k}\big(V_i(k+s+1|k)\big)$ is non-increasing with time. As $s \to \infty$, $\mathbb{E}_{k+s|k}\big(V_i(k+s+1|k)\big) \to 0$. Since $V_i(k+s+1|k) \geq \alpha_i \parallel x_i(k+s+1|k) \parallel_1$, then $\mathbb{E}_{k+s|k}\big(\parallel x_i(k+s+1|k) \parallel_1 |x(k), r_k\big) \to 0$, where $\alpha_i = \min_{\substack{p=1,2,\ldots,S,\\ j=1,2,\ldots,n}} v_i^{(j)}(k, p)$ with $v_i^{(j)}(k, p)$ being the jth element of $v_i(k, p)$. By Definition 2, the system (5) is stochastically stable.

(b) Polytopic uncertainty. The proof of the polytopic uncertainty case is similar to (a) and omitted. □

3.3. Handling constraints

In this section, a linear programming approach is presented to handle the constraints in (8).

Theorem 3. *(Handling constraints)* ***(a) Interval uncertainty.*** *If there exist constants $\hbar > 1$, $\gamma_i(k) > 0$, $\epsilon > 0$ and $\Re^n$ vectors $v_i(k, p) \succ 0, \xi_i^{(\iota)}(k, p) \prec 0, \overline{\xi}_i(k, p) \prec 0$ such that (15), (16), and*

$$v_i(k, p) \succeq \epsilon e_n, \tag{30a}$$

$$\gamma_i(k) \leq \delta\epsilon, \tag{30b}$$

$$\eta e_{m_i}^T(\underline{B}_i^T(p)\sum_{q=1}^{S}\pi_{pq}v_i(k,q)+\varrho_i(p))e_n+\delta\sum_{\iota=1}^{m_i}\xi_i^{(\iota)}(k,p)e_{m_i}^{(\iota)T}e_{m_p}\succeq 0, \tag{30c}$$

hold $\forall p\in\mathbb{S}, \forall(i,j)\in\{1,\dots,N\}\times\{1,\dots,N\}, i\neq j$, *then the constraints in (8) are handled under the control law (17).*

(b) Polytopic uncertainty. *If there exist constants* $\hbar>1$, $\gamma_i(k)>0$, $\epsilon>0$ *and vectors* $v_i(k,p)\succ 0$ *with* $v_i(k,p)\in\Re^n$, $\rho_i(p)\succ 0$ *with* $\rho_i(p)\in\Re^{m_i}$, $\xi_i^{(\iota)}(k,p)\in\Re^n$, $\overline{\xi}_i(k,p)\prec 0$ *with* $\overline{\xi}_i(k,p)\in\Re^n$ *such that (18), (16), and*

$$\begin{aligned} v_i(k,p)&\succeq\epsilon e_n,\\ \gamma_i(k)&\le\delta\epsilon,\\ \eta e_{m_i}^T\rho_i(p)e_n+\delta\textstyle\sum_{\iota=1}^{m_i}\xi_i^{(\iota)}(k,p)e_{m_i}^{(\iota)T}e_{m_p}&\succeq 0, \end{aligned} \tag{31}$$

hold $\forall l\in\{1,\dots,L\}, \forall p\in\mathbb{S}, \forall(i,j)\in\{1,\dots,N\}\times\{1,\dots,N\}, i\neq j$, *then the constraints in (8) are handled under the control law (19).*

Proof. (a) Interval uncertainty. From Theorem 1, the condition (14) holds. Then $x_i^T(k+s+1|k)v_i(k,p)\le x_i^T(k+s|k)v_i(k,p)\le\dots\le x_i^T(k|k)v_i(k,p)$. Together with (16) gives $x_i^T(k+s+1|k)v_i(k,p)\le\gamma_i(k)$. By (30a) and (30b), it derives that

$$\epsilon x_i^T(k+s+1|k)e_n\le x_i^T(k+s+1|k)v_i(k,p)\le\gamma_i(k)\le\delta\epsilon, \tag{32}$$

which verifies the validity of the constraint (8a).

Using (30c) gives

$$\eta e_n+\delta\frac{\sum_{\iota=1}^{m_i}\xi_i^{(\iota)}(k,p)e_{m_i}^{(\iota)T}}{e_{m_i}^T(\underline{B}_i^T(p)\sum_{q=1}^{S}\pi_{pq}v_i(k,q)+\varrho_i(p))}e_{m_i}\succeq 0.$$

Noting the control law in (17), it follows that $-\delta F_i^T(k,p)e_{m_i}\preceq\eta e_n$. Thus, $-\delta x_i^T(k+s|k)F_i^T(k,p)e_{m_p}\le\eta x_i^T(k+s|k)e_n$, that is, $-u_i^T(k+s|k)e_{m_i}\le\frac{\eta}{\delta}x_i^T(k+s|k)e_n$. By (32), $-u_i^T(k+s|k)e_{m_i}=\|u(k+s|k)\|_1\le\frac{\eta}{\delta}x_i^T(k+s|k)e_n\le\eta$, which handles the constraint (8b).

(b) Polytopic uncertainty. The proof is similar to that in (a) and omitted. □

To obtain the value of $\gamma_i(k)$, a linear programming can be implemented:

$$\min_{v_i(k,p),\xi_i^{(\iota)}(k,p),\overline{\xi}_i(k,p),\epsilon}\ \gamma_i(k)\ \text{ subject to (15), (16), and (30) (or, (18), (16), and (31)).} \tag{33}$$

The conditions in Theorems 1 and 3 are all linear programming. Thus, the optimization (29) and (33) can be solved in terms of linear programming.

3.4. Robust DMPC algorithm

In this section, a cone as the invariant set is established for the systems. Then, a linear programming based DMPC algorithm is proposed.

Lemma 4. *(Invariant set) Define a cone* $\Theta_i=\{x_i|x_i^T(k)v_i(k,p)\le\gamma_i(k)\}, \forall x(k)\succeq 0, \gamma_p(k)>0, \forall p\in\{1,\dots,S\}$. *Then,* Θ_i *is an invariant set of the system (5).*

By (15) and (18), the condition (14) holds. Together with (16), it is clear that Θ_i *is an invariant set of the system (5). From the proof of Theorem 2, it follows that* $\Theta_i\in\{x(k)|\mathbb{E}\{\|x(k)\|_1|x_0,r_0\}\to 0\}$.

Theorem 4. *If the optimization (33) is feasible at time instant k for the initial state $x(k)$ and initial mode r_k, then the optimization (33) is also feasible at any time instant $t \geq k$. Moreover, the DMPC controller obtained from (33) guarantees the robustly stochastic stability of the system (5) with interval/polytopic uncertainties in the mean-square sense.*

The proof of Theorem 4 is similar to that in Theorem 2 and omitted. To solving the optimization (33), an algorithm is given as follows:

Algorithm 2

Step 1: Set $F_j(0, p) = 0$ for $k = 0$. Choose a value $\hbar$ such that (33) is feasible using linear search method. Assume that the feasible solutions are $F_i^{(0)}(0, p)$. Set $F_i(0, p) = F_i^{(0)}(0, p)$ at time instant $k = 0$.

Step 2: Fix the value $\hbar$ in Step 1 and implement the linear programming (33) at the time instant $s = 1$ to obtain $F_i(1, p)$.

Step 3: Implement the linear programming (33) at the time instant $s = n$ to obtain $F_i(n, p)$. If (33) is feasible, check the convergence condition $\|F_i(n, p) - F_i(n-1, p)\|_1 \leq \varepsilon_i$, $\forall i \in \{1, \ldots, N\}$, where $\varepsilon_i > 0$ is a prescribed error. If (33) is infeasible, set $F_i(n, p) = F_i(n-1, p)$.

Step 4: Apply the control input $u_i(k+n|k) = F_i(k, p)x(k+n|k)$ to the corresponding subsystem and implement the linear programming (33) at the time instant $k = k+n+1$ by repeating Step 3.

4. Extensions to general systems

In Section 3, the DMPC of PMJSs is considered. It has also been stated that the linear approach has some advantages with respect to the quadratic approach. This section attempts to develop the approach in Section 3 for general systems. In Section 3, the interval and polytopic uncertainties in Eqs. (2) and (3) require the nonnegative property of the system matrices. Here, the nonnegative restriction is removed, that is, the system matrices in Eqs. (2) and (3) do not contain any sign restriction. Assume that the considered systems can be positively stabilized, which means that there exists a control law such that the considered systems are positive and stable.

We modify the optimization problem (10) as

$$\min_{\substack{u_i(k+s|k)\\ i=1,2,\ldots,N,s\geq 0}} \max_{\substack{[A_i(r_k)\ \ B_i(r_k)]\in\Omega_1(\text{or},\Omega_2)\\ i=1,2,\ldots,N,r_{k+s|k}\in\mathbb{S}}} J_i(k) \quad \text{subject to} \quad (5) \text{ and } (8), \tag{34}$$

with the performance index function:

$$\begin{aligned} J_i(k) = \mathbb{E}_k\Bigg\{ \sum_{s=0}^{\infty} \Big(& x_i^T(k+s|k)\varsigma(k+s|k) + u_i^{+T}(k+s|k)\varrho_i^{+}(k+s|k) + u_i^{-T}(k+s|k)\varrho_i^{-}(k+s|k) \\ & + \sum_{j=1, j\neq i}^{N} \Big(u_j^{*-T}(k+s|k)\varrho_j^{-}(k+s|k) + u_j^{*+T}(k+s|k)\varrho_j^{+}(k+s|k)\Big)\Big)\Bigg\}, \end{aligned} \tag{35}$$

where $x_i(k+s|k)$ and $u_i(k+s|k) = u_i^{+}(k+s|k) + u_i^{-}(k+s|k)$ with $u_i^{+}(k+s|k) \succeq 0$ and $u_i^{-}(k+s|k) \preceq 0$ are the state and input predicted at time instant k, $\varsigma(k+s|k) \succ 0$, $\varrho_i^{+}(k+s|k) \succ 0$, $\varrho_i^{-}(k+s|k) \prec 0$, $\varrho_j^{+}(k+s|k) \succ 0$, $\varrho_j^{-}(k+s|k) \prec 0$, and $u_j^{*}(\cdot) = u_j^{+*}(\cdot) + u_j^{-*}(\cdot)$ is the solution obtained from a previous iteration and kept fixed in the current iteration.

Remark 5. Considering the property of positive systems, a negative DMPC controller is designed in Section 3, that is, $u_i(k+s|k) \prec 0$. For general systems, the negative DMPC controller is rigorous. Therefore, the sign restriction of the controller is removed in this section. Consequently, the optimization (34) is introduced. For the DMPC of general systems, a quadratic performance index is usually used [41–43]. Here, a positive system approach is employed to deal with the DMPC problem of MJSs. Following the approach in Section 3, a linear performance index (35) is given.

The linear robust stability condition (14) is changed as:

$$
\begin{aligned}
&V_i(k+s+1|k) - V_i(k+s|k) \\
&\quad \leq -\Big(x_i^T(k+s|k)\varsigma(k+s|k) + u_i^{+T}(k+s|k)\varrho_i^{+}(k+s|k) + u_i^{-T}(k+s|k)\varrho_i^{-}(k+s|k) \\
&\quad + \sum_{j=1, j\neq i}^{N} \big(u_j^{*-T}(k+s|k)\varrho_j^{-}(k+s|k) + u_j^{*+T}(k+s|k)\varrho_j^{+}(k+s|k)\big)\Big).
\end{aligned} \tag{36}
$$

4.1. Interval uncertainty

The first section first considers the interval uncertainty case. From Section 3, it can be found that the sign of the system matrices $B_i(p)$ is key to the DMPC design. The matrices $B_i(p)$ is divided into $B_i(p) = B_i^-(p) + B_i^+(p)$, where $B_i^-(p) \preceq 0$ and $B_i^+(p) \succeq 0$ consisting of all non-positive and nonnegative elements of $B_i(p)$, respectively. Then, denote $\underline{B}_i(p) = \underline{B}_i^-(p) + \underline{B}_i^+(p)$ and $\overline{B}_i(p) = \overline{B}_i^-(p) + \overline{B}_i^+(p)$, where $\underline{B}_i^-(p) \preceq 0, \underline{B}_i^+(p) \succeq 0, \overline{B}_i^-(p) \preceq 0, \overline{B}_i^+(p) \succeq 0$ and $\underline{B}_i^-(p), \underline{B}_i^+(p)$ and $\overline{B}_i^-(p), \overline{B}_i^+(p)$ are the corresponding lower and upper bounds, respectively. For the cases $B_i^-(p) = 0$ and $B_i^+(p) = 0$, the results are trivial and similar to Section 3. Therefore, only the cases $B_i^-(p) \neq 0$ and $B_i^+(p) \neq 0$ are considered. Accordingly, the controller gain is given as: $F_i(k,p) = F_i^-(k,p) + F_i^+(k,p)$, where $F_i^-(k,p) \preceq 0$ and $F_i^+(k,p) \succeq 0$.

Theorem 5 (Controller design)**.** *If there exist constants $\hbar_1 > 1$, $\hbar_2 > 1$, $\gamma_i(k) > 0$ and $\Re^n$ vectors $v_i(k,p) \succ 0$, $\xi_i^{(\iota)+}(k,p) \succ 0$, $\xi_i^{+}(k,p) \succ 0$, $\xi_i^{(\iota)-}(k,p) \prec 0$, $\xi_i^{-}(k,p) \prec 0$ such that*

$$
\overline{A}_i^{*T}(p)\sum_{q=1}^{S}\pi_{pq}v_i(k,q) + \hbar_2\xi_i^{+}(k,p) + \xi_i^{-}(k,p) - v_i(k,p) + \varsigma^*(p) \prec 0, \tag{37a}
$$

$$
\begin{aligned}
&\underline{A}_i(p)e_{m_i}^T(\underline{B}_i^T(p)\sum_{q=1}^{S}\pi_{pq}v_i(k,q) + \varrho_i^-(p)) + \frac{1}{\hbar_1}\sum_{j=1, j\neq i}^{N}\underline{B}_j^+(p)\sum_{\iota=1}^{m_j}e_{m_j}^{(\iota)}\xi_j^{(\iota)+T}(k,p) \\
&+ \hbar_1\sum_{j=1, j\neq i}^{N}\underline{B}_j^-(p)\sum_{\iota=1}^{m_j}e_{m_j}^{(\iota)}\xi_j^{(\iota)+T}(k,p) + \hbar_1\sum_{j=1, j\neq i}^{N}\overline{B}_j^+(p)\sum_{\iota=1}^{m_j}e_{m_j}^{(\iota)}\xi_j^{(\iota)-T}(k,p) \\
&+ \frac{1}{\hbar_1}\sum_{j=1, j\neq i}^{N}\overline{B}_j^-(p)\sum_{\iota=1}^{m_j}e_{m_j}^{(\iota)}\xi_j^{(\iota)-T}(k,p) + \underline{B}_i(p)\sum_{\iota=1}^{m_j}e_{m_j}^{(\iota)}\xi_j^{(\iota)+T}(k,p) \\
&+ \overline{B}_i(p)\sum_{\iota=1}^{m_j}e_{m_j}^{(\iota)}\xi_j^{(\iota)-T}(k,p) \succeq 0,
\end{aligned} \tag{37b}
$$

$$
e_{m_i}^T(\underline{B}_i^T(p)\textstyle\sum_{q=1}^{S}\pi_{pq}v_i(k,q) + \varrho_i^-(p)) \leq \hbar_1 e_{m_j}^T(\underline{B}_j^T(p)\sum_{q=1}^{S}\pi_{pq}v_j(k,q) + \varrho_j^-(p)), \tag{37c}
$$

$$e_{m_i}^T(\overline{B}_i^T(p)\textstyle\sum_{q=1}^S \pi_{pq}v_i(k,q)+\varrho_i^-(p)) \leq \hbar_2 e_{m_i}^T(\underline{B}_i^T(p)\sum_{q=1}^S \pi_{pq}v_i(k,q)+\varrho_i^-(p)), \tag{37d}$$

$$\underline{B}_i^T(p)\textstyle\sum_{q=1}^S \pi_{pq}v_i(k,q)+\varrho_i^-(p)\succ 0, \tag{37e}$$

$$\xi_i^{(\iota)+}(k,p)\preceq \xi_i^+(k,p), \xi_i^{(\iota)-}(k,p)\preceq \xi_i^-(k,p),\ \ \iota=1,2,\ldots,m_i, \tag{37f}$$

and (16) hold $\forall p\in\mathbb{S}, \forall(i,j)\in\{1,\ldots,N\}\times\{1,\ldots,N\}, i\neq j$, *then under the control law* $u_i(k+s|k)=F_i(k,p)x_i(k+s|k)=(F_i^+(k,p)+F_i^-(k,p))x_i(k+s|k)$ *with*

$$\begin{aligned} F_i^+(k,p)&=\frac{\sum_{\iota=1}^{m_i} e_{m_i}^{(\iota)}\xi_i^{(\iota)+T}(k,p)}{e_{m_i}^T(\underline{B}_i^T(p)\sum_{q=1}^S \pi_{pq}v_i(k,q)+\varrho_i^-(p))},\\ F_i^-(k,p)&=\frac{\sum_{\iota=1}^{m_i} e_{m_i}^{(\iota)}\xi_i^{(\iota)-T}(k,p)}{e_{m_i}^T(\underline{B}_i^T(p)\sum_{q=1}^S \pi_{pq}v_i(k,q)+\varrho_i^-(p))}, \end{aligned} \tag{38}$$

the interval uncertain system (5) is positive and satisfies the condition (36), or there exist constants $1>\hbar_1>0$, $\gamma_i(k)>0$ *and* $\Re^n$ *vectors* $v_i(k,p)\succ 0$, $\xi_i^{(\iota)+}(k,p)\succ 0$, $\xi_i^+(k,p)\succ 0$, $\xi_i^{(\iota)-}(k,p)\prec 0$, $\xi_i^-(k,p)\prec 0$ *such that*

$$\begin{aligned} &\overline{A}_i^{*T}(p)\sum_{q=1}^S \pi_{pq}v_i(k,q)+\xi_i^+(k,p)+\xi_i^-(k,p)-v_i(k,p)+\varsigma^*(p)\prec 0,\\ &\underline{A}_i(p)e_{m_i}^T(\underline{B}_i^T(p)\sum_{q=1}^S \pi_{pq}v_i(k,q)+\varrho_i^-(p))+\hbar_1\sum_{j=1,j\neq i}^N \underline{B}_j^+(p)\sum_{\iota=1}^{m_j} e_{m_j}^{(\iota)}\xi_j^{(\iota)-T}(k,p)\\ &+\frac{1}{\hbar_1}\sum_{j=1,j\neq i}^N \underline{B}_j^-(p)\sum_{\iota=1}^{m_j} e_{m_j}^{(\iota)}\xi_j^{(\iota)-T}(k,p)+\frac{1}{\hbar_1}\sum_{j=1,j\neq i}^N \overline{B}_j^+(p)\sum_{\iota=1}^{m_j} e_{m_j}^{(\iota)}\xi_j^{(\iota)+T}(k,p)\\ &+\hbar_1\sum_{j=1,j\neq i}^N \overline{B}_j^-(p)\sum_{\iota=1}^{m_j} e_{m_j}^{(\iota)}\xi_j^{(\iota)+T}(k,p)+\underline{B}_i(p)\sum_{\iota=1}^{m_j} e_{m_j}^{(\iota)}\xi_j^{(\iota)+T}(k,p)\\ &+\overline{B}_i(p)\sum_{\iota=1}^{m_j} e_{m_j}^{(\iota)}\xi_j^{(\iota)-T}(k,p)\succeq 0,\\ &e_{m_i}^T(\underline{B}_i^T(p)\sum_{q=1}^S \pi_{pq}v_i(k,q)+\varrho_i^-(p))\leq \hbar_1 e_{m_j}^T(\underline{B}_j^T(p)\sum_{q=1}^S \pi_{pq}v_j(k,q)+\varrho_j^-(p)),\\ &\overline{B}_i^T(p)\sum_{q=1}^S \pi_{pq}v_i(k,q)+\varrho_i^-(p)\prec 0,\\ &\xi_i^{(\iota)+}(k,p)\preceq \xi_i^+(k,p), \xi_i^{(\iota)-}(k,p)\preceq \xi_i^-(k,p),\ \ \iota=1,2,\ldots,m_i, \end{aligned} \tag{39}$$

and (16) hold $\forall p\in\mathbb{S}, \forall(i,j)\in\{1,\ldots,N\}\times\{1,\ldots,N\}, i\neq j$, *then under the control law* $u_i(k+s|k)=F_i(k,p)x_i(k+s|k)=(F_i^+(k,p)+F_i^-(k,p))x_i(k+s|k)$ *with*

$$F_i^+(k,p)=\frac{\sum_{\iota=1}^{m_i} e_{m_i}^{(\iota)}\xi_i^{(\iota)-T}(k,p)}{e_{m_i}^T(\underline{B}_i^T(p)\sum_{q=1}^S \pi_{pq}v_i(k,q)+\varrho_i^-(p))},$$

$$F_i^-(k,p) = \frac{\sum_{\iota=1}^{m_i} e_{m_i}^{(\iota)} \xi_i^{(\iota)+T}(k,p)}{e_{m_i}^T(\underline{B}_i^T(p)\sum_{q=1}^S \pi_{pq} v_i(k,q) + \varrho_i^-(p))}, \tag{40}$$

the interval uncertain system (5) is positive and satisfies the condition (36), where

$$\overline{A}_i^*(p) = \overline{A}_i(p) + \sum_{j=1, j\neq i}^N (\overline{B}_j(p)F_j^{+*}(k,p) + \underline{B}_j(p)F_j^{-*}(k,p)).$$

Proof. Consider the validity of Theorem 5 under the condition (37). The proof of the condition (38) is similar and omitted. By (37e), it follows that $e_{m_i}^T(\underline{B}_i^T(p)\sum_{q=1}^S \pi_{pq} v_i(k,q) + \varrho_i^-(p)) > 0$. By (37c), it is easy to obtain

$$\begin{aligned}
&\frac{\hbar_1}{e_{m_i}^T(\underline{B}_i^T(p)\sum_{q=1}^S \pi_{pq} v_i(k,q) + \varrho_i^-(p))} \geq \frac{1}{e_{m_j}^T(\underline{B}_j^T(p)\sum_{q=1}^S \pi_{pq} v_j(k,q) + \varrho_j^-(p))} \\
&\geq \frac{1}{\hbar_1 e_{m_i}^T(\underline{B}_i^T(p)\sum_{q=1}^S \pi_{pq} v_i(k,q) + \varrho_i^-(p))}.
\end{aligned} \tag{41}$$

Together with $\xi_j^{(\iota)+}(k,p) \succ 0, \xi_j^{(\iota)-}(k,p) \prec 0$, and (6) gives

$$\begin{aligned}
&A_i(p) + \sum_{j=1, j\neq i}^N B_j(p)F_j(k,p) + B_i(p)F_i(k,p) \\
&= A_i(p) + \sum_{j=1, j\neq i}^N B_j(p)(F_j^+(k,p) + F_j^-(k,p)) + B_i(p)(F_i^+(k,p) + F_i^-(k,p)) \\
&\succeq \underline{A}_i(p) + \sum_{j=1, j\neq i}^N \Big(\underline{B}_j(p)F_j^+(k,p) + \overline{B}_j(p)F_j^-(k,p)\Big) + \Big(\underline{B}_i(p)F_i^+(k,p) + \overline{B}_i(p)F_i^-(k,p)\Big).
\end{aligned} \tag{42}$$

By Eqs. (38) and (41),

$$\begin{aligned}
\underline{B}_j(p)F_j^+(k,p) \succeq{}& \underline{B}_j^+(p)\frac{\sum_{\iota=1}^{m_j} e_{m_j}^{(\iota)} \xi_j^{(\iota)+T}(k,p)}{\hbar_1 e_{m_i}^T(\underline{B}_i^T(p)\sum_{q=1}^S \pi_{pq} v_j(k,q) + \varrho_i^-(p))} \\
&+ \underline{B}_j^-(p)\frac{\hbar_1 \sum_{\iota=1}^{m_j} e_{m_j}^{(\iota)} \xi_j^{(\iota)+T}(k,p)}{e_{m_i}^T(\underline{B}_i^T(p)\sum_{q=1}^S \pi_{pq} v_i(k,q) + \varrho_i^-(p))}, \\
\overline{B}_j(p)F_j^-(k,p) \succeq{}& \overline{B}_j^+(p)\frac{\hbar_1 \sum_{\iota=1}^{m_j} e_{m_j}^{(\iota)} \xi_j^{(\iota)-T}(k,p)}{e_{m_i}^T(\underline{B}_i^T(p)\sum_{q=1}^S \pi_{pq} v_i(k,q) + \varrho_i^-(p))} \\
&+ \overline{B}_j^-(p)\frac{\sum_{\iota=1}^{m_j} e_{m_j}^{(\iota)} \xi_j^{(\iota)-T}(k,p)}{\hbar_1 e_{m_i}^T(\underline{B}_i^T(p)\sum_{q=1}^S \pi_{pq} v_i(k,q) + \varrho_i^-(p))}.
\end{aligned} \tag{43}$$

Substituting (43) into (42) yields that

$$A_i(p) + \sum_{j=1, j\neq i}^N B_j(p)F_j(k,p) + B_i(p)F_i(k,p)$$

$$
\begin{aligned}
&\succeq \frac{1}{e_{m_i}^T(\underline{B}_i^T(p)\sum_{q=1}^S \pi_{pq}v_i(k,q)+\varrho_i(p))}\Big(e_{m_i}^T(\underline{B}_i^T(p)\sum_{q=1}^S \pi_{pq}v_i(k,q)+\varrho_i(p))\underline{A}_i(p)\\
&+\frac{1}{\hbar_1}\sum_{j=1,j\neq i}^N \underline{B}_j^+(p)\sum_{\iota=1}^{m_j} e_{m_j}^{(\iota)}\xi_j^{(\iota)+T}(k,p)+\hbar_1\sum_{j=1,j\neq i}^N \underline{B}_j^-(p)\sum_{\iota=1}^{m_j} e_{m_j}^{(\iota)}\xi_j^{(\iota)+T}(k,p)\\
&+\hbar_1\sum_{j=1,j\neq i}^N \overline{B}_j^+(p)\sum_{\iota=1}^{m_j} e_{m_j}^{(\iota)}\xi_j^{(\iota)-T}(k,p)+\frac{1}{\hbar_1}\sum_{j=1,j\neq i}^N \overline{B}_j^-(p)\sum_{\iota=1}^{m_j} e_{m_j}^{(\iota)}\xi_j^{(\iota)-T}(k,p)\\
&+\underline{B}_i(p)\sum_{\iota=1}^{m_j} e_{m_j}^{(\iota)}\xi_j^{(\iota)+T}(k,p)+\overline{B}_i(p)\sum_{\iota=1}^{m_j} e_{m_j}^{(\iota)}\xi_j^{(\iota)-T}(k,p)\Big).
\end{aligned}
\tag{44}
$$

From Eq. (37b), it is clear that $A_i(p)+\sum_{j=1,j\neq i}^N B_j(p)F_j(k,p)+B_i(p)F_i(k,p)\succeq 0$, which implies that the pth mode of the interval uncertain system (5) is positive by Lemma 1. Thus, the interval uncertain system (5) is positive by Lemma 2.

Next, consider the validity of the condition (36). Similar to (14), the condition (36) is equivalent to:

$$
\begin{aligned}
&x_i^T(k+s|k)\Big(\big(A_i^*(p)+B_i(p)F_i(k,p)\big)^T\sum_{q=1}^S \pi_{pq}v_i(k,q)-v_i(k,p)\\
&+\varsigma^*(p)+F_i^{+T}(k,p)\varrho_i^+(p)+F_i^{-T}(k,p)\varrho_i^-(p)\Big)\preceq 0,
\end{aligned}
$$

where $\varsigma^*(p)=\varsigma(p)+\sum_{j=1,j\neq i}^N(F_j^{+*T}(k,p)\varrho_j^+(p)+F_j^{-*T}(k,p)\varrho_j^-(p))$. By (37d) and (38),

$$
\begin{aligned}
&F_i^T(k,p)B_i^T(p)\sum_{q=1}^S \pi_{pq}v_i(k,q)+F_i^{+T}(k,p)\varrho_i^+(p)+F_i^{-T}(k,p)\varrho_i^-(p)\\
&=\frac{\sum_{\iota=1}^{m_i}\xi_i^{(\iota)+}(k,p)e_{m_i}^{(\iota)T}}{e_{m_i}^T(\underline{B}_i^T(p)\sum_{q=1}^S \pi_{pq}v_i(k,q)+\varrho_i^-(p))}(B_i^T(p)\sum_{q=1}^S \pi_{pq}v_i(k,q)+\varrho_i^+(p))\\
&+\frac{\sum_{\iota=1}^{m_i}\xi_i^{(\iota)-}(k,p)e_{m_i}^{(\iota)T}}{e_{m_i}^T(\underline{B}_i^T(p)\sum_{q=1}^S \pi_{pq}v_i(k,q)+\varrho_i^-(p))}(B_i^T(p)\sum_{q=1}^S \pi_{pq}v_i(k,q)+\varrho_i^-(p))\\
&\preceq\frac{\sum_{\iota=1}^{m_i}\xi_i^{(\iota)+}(k,p)e_{m_i}^{(\iota)T}}{e_{m_i}^T(\underline{B}_i^T(p)\sum_{q=1}^S \pi_{pq}v_i(k,q)+\varrho_i^-(p))}(\overline{B}_i^T(p)\sum_{q=1}^S \pi_{pq}v_i(k,q)+\varrho_i^+(p))\\
&+\frac{\sum_{\iota=1}^{m_i}\xi_i^{(\iota)-}(k,p)e_{m_i}^{(\iota)T}}{e_{m_i}^T(\underline{B}_i^T(p)\sum_{q=1}^S \pi_{pq}v_i(k,q)+\varrho_i^-(p))}(\underline{B}_i^T(p)\sum_{q=1}^S \pi_{pq}v_i(k,q)+\varrho_i^-(p))\\
&\preceq\hbar_2\xi_i^+(k,p)+\xi_i^-(k,p).
\end{aligned}
\tag{45}
$$

From (6) and (45),

$$
\big(A_i^*(p)+B_i(p)F_i(k,p)\big)^T\sum_{q=1}^S \pi_{pq}v_i(k,q)-v_i(k,p)
$$

$$+\varsigma^{*}(p)+F_i^{+T}(k,p)\varrho_i^{+}(p)+F_i^{-T}(k,p)\varrho_i^{-}(p)$$
$$\preceq \overline{A}_i^{*T}(p)\sum_{q=1}^{S}\pi_{pq}v_i(k,q)+\hbar_2\xi_i^{+}(k,p)+\xi_i^{-}(k,p)-v_i(k,p)+\varsigma^{*}(p). \tag{46}$$

Combining the fact $x_i^T(k+s|k)\succeq 0$, (37a), and (46) concludes that the condition (36) holds. □

Remark 6. In Theorem 1, the open-loop interval uncertain system (5) is assumed to be positive. The condition (15b) is presented to guarantee the positivity of the corresponding closed-loop system. In Theorem 5, a positive system approach is used for the DMPC of general systems (non-positive open-loop systems). To guarantee the positivity of the system (5), the condition (37b) is introduced. Noting the controller gain matrices in (17) and (38), the former is required to be negative whereas the latter is not. Assume that Theorem 5 employs a similar controller to (17), then the corresponding positivity condition has a similar form to (15b). On the other hand, the DMPC design approach in Theorem 5 can be developed for Theorem 1 and remove the sign restriction of the DMPC controller gain matrix in Theorem 1.

Theorem 6 (Handling constraints). *If there exist constants $\hbar_1>1$, $\hbar_2>1$, $\gamma_i(k)>0$, $\epsilon>0$ and $\Re^n$ vectors $v_i(k,p)\succ 0$, $\xi_i^{(\iota)+}(k,p)\succ 0$, $\xi_i^{+}(k,p)\succ 0$, $\xi_i^{(\iota)-}(k,p)\prec 0$, $\xi_i^{-}(k,p)\prec 0$ such that (37), (16), and*

$$v_i(k,p)\succeq \epsilon e_n, \tag{47a}$$

$$\gamma_i(k)\leq \delta\epsilon, \tag{47b}$$

$$\delta\sum_{\iota=1}^{m_i}\xi_i^{(\iota)+}(k,p)e_{m_i}^{(\iota)T}e_{m_p}-\delta\sum_{\iota=1}^{m_i}\xi_i^{(\iota)-}(k,p)e_{m_i}^{(\iota)T}e_{m_p}-\eta e_{m_i}^T(\underline{B}_i^T(p)\sum_{q=1}^{S}\pi_{pq}v_i(k,q)$$
$$+\varrho_i^{-}(p))e_n\preceq 0, \tag{47c}$$

hold $\forall p\in\mathbb{S}$, $\forall(i,j)\in\{1,\dots,N\}\times\{1,\dots,N\}$, $i\neq j$, then the constraints in (8) are handled under the control law (38), or there exist constants $1>\hbar_1>0$, $\gamma_i(k)>0$, $\epsilon>0$ and vectors $v_i(k,p)\succ 0$, $\xi_i^{(\iota)+}(k,p)\succ 0$, $\xi_i^{+}(k,p)\succ 0$, $\xi_i^{(\iota)-}(k,p)\prec 0$, $\xi_i^{-}(k,p)\prec 0$ such that (39), (16), and

$$v_i(k,p)\succeq \epsilon e_n,$$
$$\gamma_i(k)\leq \delta\epsilon,$$
$$\delta\sum_{\iota=1}^{m_i}\xi_i^{(\iota)-}(k,p)e_{m_i}^{(\iota)T}e_{m_p}-\delta\sum_{\iota=1}^{m_i}\xi_i^{(\iota)+}(k,p)e_{m_i}^{(\iota)T}e_{m_p}$$
$$-\eta e_{m_i}^T(\underline{B}_i^T(p)\sum_{q=1}^{S}\pi_{pq}v_i(k,q)+\varrho_i^{-}(p))e_n\succeq 0, \tag{48}$$

hold $\forall p\in\mathbb{S}$, $\forall(i,j)\in\{1,\dots,N\}\times\{1,\dots,N\}$, $i\neq j$, then the constraints in Eq. (8) are handled under the control law (40).

Proof. From Eqs. (47c) and (38), $\delta F_i^{+T}(k,p)e_{m_p} - \delta F_i^{-T}(k,p)e_{m_p} - \eta e_n \preceq 0$. Thus, $\delta u_i^{+T}(k+s|k)e_{m_p} - \delta u_i^{-T}(k+s|k)e_{m_p} \le \eta x^T(k+s|k)e_n$, and consequently,

$$\|u_i(k+s|k)\|_1 \le \|u_i^+(k+s|k)\|_1 + \|u_i^-(k+s|k)\|_1 = u_i^{+T}(k+s|k)e_{m_p} - u_i^{-T}(k+s|k)e_{m_p} \le \eta.$$

The proof of the second case is similar to the first case and is omitted. □

Theorems 5–6 have developed the DMPC design approach of interval positive systems to general systems. The feasibility and robustly stochastic stability of the systems can be given using similar methods in Theorems 3–4Theorems 3 and 4.

4.2. Polytopic uncertainty

This section further develops the approach in Section 3 for the polytopic case. Similar to Section 4.1, the sign restriction of the system matrices is removed. Denote $B_i^{(l)}(p) = B_i^{(l)-}(p) + B_i^{(l)+}(p)$, where $B_i^{(l)-}(p) \preceq 0$ and $B_i^{(l)+}(p) \succeq 0$.

Theorem 7 (Controller design). *If there exist constants $\hbar_1 > 1$, $\hbar_2 > 1$, $\gamma_i(k) > 0$ and vectors $v_i(k,p) \succ 0$, $\xi_i^{(\iota)+}(k,p) \succ 0$, $\xi_i^+(k,p) \succ 0$, $\xi_i^{(\iota)-}(k,p) \prec 0$, $\xi_i^-(k,p) \prec 0$, $\rho_i(p) \succ 0$ such that*

$$A_i^{(l)*T}(p)\sum_{q=1}^{S}\pi_{pq}v_i(k,q) + \hbar_2\xi_i^+(k,p) + \xi_i^-(k,p) - v_i(k,p) + \varsigma^*(p) \prec 0, \tag{49a}$$

$$\begin{aligned}
&A_i^{(l)}(p)e_{m_i}^T\rho_i(p) + \frac{1}{\hbar_1}\sum_{j=1,j\neq i}^{N}B_j^{(l)+}(p)\sum_{\iota=1}^{m_j}e_{m_j}^{(\iota)}\xi_j^{(\iota)+T}(k,p) + \hbar_1\sum_{j=1,j\neq i}^{N}B_j^{(l)+}(p)\sum_{\iota=1}^{m_j}e_{m_j}^{(\iota)}\xi_j^{(\iota)-T}(k,p)\\
&+\hbar_1\sum_{j=1,j\neq i}^{N}B_j^{(l)-}(p)\sum_{\iota=1}^{m_j}e_{m_j}^{(\iota)}\xi_j^{(\iota)+T}(k,p) + \hbar_1\sum_{j=1,j\neq i}^{N}B_j^{(l)-}(p)\sum_{\iota=1}^{m_j}e_{m_j}^{(\iota)}\xi_j^{(\iota)-T}(k,p)\\
&+B_i^{(l)}(p)\sum_{\iota=1}^{m_i}e_{m_i}^{(\iota)}\xi_i^{(\iota)+T}(k,p) + B_i^{(l)}(p)\sum_{\iota=1}^{m_i}e_{m_i}^{(\iota)}\xi_i^{(\iota)-T}(k,p) \succeq 0,
\end{aligned} \tag{49b}$$

$$e_{m_i}^T\rho_i(p) \le \hbar_1 e_{m_j}^T\rho_j(p), \tag{49c}$$

$$B_i^{(l)T}(p)\textstyle\sum_{q=1}^{S}\pi_{pq}v_i(k,q) + \varrho_i^-(p) \succ \rho_i(p), \tag{49d}$$

$$B_i^{(l)T}(p)\textstyle\sum_{q=1}^{S}\pi_{pq}v_i(k,q) + \varrho_i^+(p) \prec \hbar_2\rho_i(p), \tag{49e}$$

$$\xi_i^{(\iota)+}(k,p) \preceq \xi_i^+(k,p), \xi_i^{(\iota)-}(k,p) \preceq \xi_i^-(k,p), \ \ \iota = 1,2,\ldots,m_i, \tag{49f}$$

and (16) hold $\forall p \in \mathbb{S}, \forall l \in \{1,\ldots,L\}, \forall (i,j) \in \{1,\ldots,N\}\times\{1,\ldots,N\}, i \neq j$, then under the control law $u_i(k+s|k) = F_i(k,p)x_i(k+s|k) = (F_i^+(k,p) + F_i^-(k,p))x_i(k+s|k)$ with

$$F_i^+(k,p) = \frac{\sum_{\iota=1}^{m_i}e_{m_i}^{(\iota)}\xi_i^{(\iota)+T}(k,p)}{e_{m_i}^T\rho_i(p)}, \ \ F_i^-(k,p) = \frac{\sum_{\iota=1}^{m_i}e_{m_i}^{(\iota)}\xi_i^{(\iota)-T}(k,p)}{e_{m_i}^T\rho_i(p)}, \tag{50}$$

the polytopic system (5) is positive and satisfies the condition (36), or there exist constants $1 > \hbar_1 > 0$, $\gamma_i(k) > 0$ *and vectors* $v_i(k, p) \succ 0$, $\xi_i^{(\iota)+}(k, p) \succ 0$, $\xi_i^{+}(k, p) \succ 0$, $\xi_i^{(\iota)-}(k, p) \prec 0$, $\xi_i^{-}(k, p) \prec 0$, $\rho_i(p) \prec 0$ *such that*

$$
\begin{aligned}
&A_i^{(l)*T}(p)\sum_{q=1}^{S}\pi_{pq}v_i(k,q)+\xi_i^{+}(k,p)+\xi_i^{-}(k,p)-v_i(k,p)+\varsigma^*(p)\prec 0,\\
&A_i^{(l)}(p)e_{m_i}^T\rho_i(p)+\hbar_1\sum_{j=1,j\neq i}^{N}B_j^{(l)+}(p)\sum_{\iota=1}^{m_j}e_{m_j}^{(\iota)}\xi_j^{(\iota)-T}(k,p)\\
&+\frac{1}{\hbar_1}\sum_{j=1,j\neq i}^{N}B_j^{(l)+}(p)\sum_{\iota=1}^{m_j}e_{m_j}^{(\iota)}\xi_j^{(\iota)+T}(k,p)\\
&+\frac{1}{\hbar_1}\sum_{j=1,j\neq i}^{N}B_j^{(l)-}(p)\sum_{\iota=1}^{m_j}e_{m_j}^{(\iota)}\xi_j^{(\iota)-T}(k,p)+\frac{1}{\hbar_1}\sum_{j=1,j\neq i}^{N}B_j^{(l)-}(p)\sum_{\iota=1}^{m_j}e_{m_j}^{(\iota)}\xi_j^{(\iota)+T}(k,p)\\
&+B_i^{(l)}(p)\sum_{\iota=1}^{m_i}e_{m_i}^{(\iota)}\xi_i^{(\iota)+T}(k,p)+B_i^{(l)}(p)\sum_{\iota=1}^{m_i}e_{m_i}^{(\iota)}\xi_i^{(\iota)-T}(k,p)\preceq 0,\\
&e_{m_i}^T\rho_i(p)\leq\hbar_1e_{m_j}^T\rho_j(p),\\
&B_i^{(l)T}(p)\sum_{q=1}^{S}\pi_{pq}v_i(k,q)+\varrho_i^{+}(p)\prec\rho_i(p),\\
&\xi_i^{(\iota)+}(k,p)\preceq\xi_i^{+}(k,p),\xi_i^{(\iota)-}(k,p)\preceq\xi_i^{-}(k,p),\ \ \iota=1,2,\dots,m_i,
\end{aligned}
\tag{51}
$$

and (16) hold $\forall p\in\mathbb{S}, \forall l\in\{1,\dots,L\}, \forall(i,j)\in\{1,\dots,N\}\times\{1,\dots,N\}, i\neq j$, *then under the control law* $u_i(k+s|k) = F_i(k,p)x_i(k+s|k) = (F_i^{+}(k,p)+F_i^{-}(k,p))x_i(k+s|k)$ *with*

$$
F_i^{+}(k,p)=\frac{\sum_{\iota=1}^{m_i}e_{m_i}^{(\iota)}\xi_i^{(\iota)-T}(k,p)}{e_{m_i}^T\rho_i(p)},\ \ F_i^{-}(k,p)=\frac{\sum_{\iota=1}^{m_i}e_{m_i}^{(\iota)}\xi_i^{(\iota)+T}(k,p)}{e_{m_i}^T\rho_i(p)},
\tag{52}
$$

the polytopic system (5) is positive and satisfies the condition (36), where $A_i^{(l)*}(p) = A_i^{(l)}(p) + \sum_{j=1,j\neq i}^{N}B_j^{(l)}(p)F_j^*(k,p)$.

Proof. Consider the proof of the condition (49). By (49c), it follows that $\frac{\hbar}{e_{m_i}^T\rho_i(p)} \geq \frac{1}{e_{m_j}^T\rho_j(p)} \geq \frac{1}{\hbar e_{m_i}^T\rho_i(p)}$. Together with (7) and (50) gives

$$
\begin{aligned}
&A_i^{(l)}(p)+\sum_{j=1,j\neq i}^{N}B_j^{(l)}(p)F_j(k,p)+B_i^{(l)}(p)F_i(k,p)\\
&\succeq\frac{1}{e_{m_i}^T\rho_i(p)}\Big(A_i^{(l)}(p)+\frac{1}{\hbar_1}\sum_{j=1,j\neq i}^{N}B_j^{(l)+}(p)\sum_{\iota=1}^{m_j}e_{m_j}^{(\iota)}\xi_j^{(\iota)+T}(k,p)\\
&+\hbar_1\sum_{j=1,j\neq i}^{N}B_j^{(l)+}(p)\sum_{\iota=1}^{m_j}e_{m_j}^{(\iota)}\xi_j^{(\iota)-T}(k,p)
\end{aligned}
$$

$$+\hbar_1 \sum_{j=1,j\neq i}^{N} B_j^{(l)-}(p) \sum_{\iota=1}^{m_j} e_{m_j}^{(\iota)} \xi_j^{(\iota)+T}(k,p) + \hbar_1 \sum_{j=1,j\neq i}^{N} B_j^{(l)-}(p) \sum_{\iota=1}^{m_j} e_{m_j}^{(\iota)} \xi_j^{(\iota)-T}(k,p)$$
$$+ B_i^{(l)}(p) \sum_{\iota=1}^{m_i} e_{m_i}^{(\iota)} \xi_i^{(\iota)+T}(k,p) + B_i^{(l)}(p) \sum_{\iota=1}^{m_i} e_{m_i}^{(\iota)} \xi_i^{(\iota)-T}(k,p)\Big).$$

By (49b), $A_i^{(l)}(p) + \sum_{j=1,j\neq i}^{N} B_j^{(l)}(p)F_j(k,p) + B_i^{(l)}(p)F_i(k,p) \succeq 0$. Thus, the system (26) is positive by Lemma 1.

Next, the validity of the condition (36) is given. By (49d) and (49e),

$$F_i^T(k,p)B_i^T(p)\sum_{q=1}^{S}\pi_{pq}v_i(k,q) + F_i^{+T}(k,p)\varrho_i^+(p) + F_i^{-T}(k,p)\varrho_i^-(p)$$
$$= \sum_{l=1}^{L}\lambda_l\Big(F_i^{+T}(k,p)\big(B_i^{(l)T}(p)\sum_{q=1}^{S}\pi_{pq}v_i(k,q) + \varrho_i^+(p)\big)$$
$$+ F_i^{-T}(k,p)\big(B_i^{(l)T}(p)\sum_{q=1}^{S}\pi_{pq}v_i(k,q) + \varrho_i^-(p)\big)\Big)$$
$$= \sum_{l=1}^{L}\lambda_l\Big(\frac{\sum_{\iota=1}^{m_i}\xi_i^{(\iota)+}(k,p)e_{m_i}^{(\iota)T}}{e_{m_i}^T\rho_i(p)}(B_i^{(l)T}(p)\sum_{q=1}^{S}\pi_{pq}v_i(k,q) + \varrho_i^+(p))$$
$$+ \frac{\sum_{\iota=1}^{m_i}\xi_i^{(\iota)-}(k,p)e_{m_i}^{(\iota)T}}{e_{m_i}^T\rho_i(p)}(B_i^{(l)T}(p)\sum_{q=1}^{S}\pi_{pq}v_i(k,q) + \varrho_i^-(p))\Big)$$
$$\preceq \sum_{l=1}^{L}\lambda_l\Big(\hbar_2\xi_i^+(k,p) + \xi_i^-(k,p)\Big).$$

Thus,

$$\big(A_i^*(p) + B_i(p)F_i(k,p)\big)^T\sum_{q=1}^{S}\pi_{pq}v_i(k,q) - v_i(k,p)$$
$$+ \varsigma^*(p) + F_i^{+T}(k,p)\varrho_i^+(p) + F_i^{-T}(k,p)\varrho_i^-(p)$$
$$= \sum_{l=1}^{L}\lambda_l\Big(\big(A_i^{(l)*}(p) + B_i^{(l)}(p)F_i(k,p)\big)^T\sum_{q=1}^{S}\pi_{pq}v_i(k,q) - v_i(k,p)$$
$$+ \varsigma^*(p) + F_i^{+T}(k,p)\varrho_i^+(p) + F_i^{-T}(k,p)\varrho_i^-(p)\Big)$$
$$\preceq \sum_{l=1}^{L}\lambda_l\Big(A_i^{(l)*T}(p)\sum_{q=1}^{S}\pi_{pq}v_i(k,q) + \hbar_2\xi_i^+(k,p) + \xi_i^-(k,p) - v_i(k,p) + \varsigma^*(p)\Big).$$

By Eq. (49a), it follows that the condition (36) holds. □

Theorem 8 (Handling constraints)**.** *If there exist constants* $\hbar_1 > 1$, $\hbar_2 > 1$, $\gamma_i(k) > 0$, $\epsilon > 0$ *and vectors* $v_i(k,p) \succ 0$, $\xi_i^{(\iota)+}(k,p) \succ 0$, $\xi_i^+(k,p) \succ 0$, $\xi_i^{(\iota)-}(k,p) \prec 0$, $\xi_i^-(k,p) \prec 0$, $\rho_i(p) \succ 0$

such that Eqs. (49), (16), and

$$\begin{array}{c} v_i(k,p) \succeq \epsilon e_n, \\ \gamma_i(k) \leq \delta\epsilon, \\ \delta \sum_{\iota=1}^{m_i} \xi_i^{(\iota)+}(k,p) e_{m_i}^{(\iota)T} e_{m_p} - \delta \sum_{\iota=1}^{m_i} \xi_i^{(\iota)-}(k,p) e_{m_i}^{(\iota)T} e_{m_p} - \eta e_{m_i}^T \rho_i(p) e_n \preceq 0, \end{array} \tag{53}$$

hold $\forall p \in \mathbb{S}, \forall (i,j) \in \{1,\ldots,N\} \times \{1,\ldots,N\}, i \neq j$, *then the constraints in Eq. (8) are handled under the control law (50), or there exist constants* $1 > \hbar_1 > 0$, $\gamma_i(k) > 0$, $\epsilon > 0$ *and vectors* $v_i(k,p) \succ 0$, $\xi_i^{(\iota)+}(k,p) \succ 0$, $\xi_i^{+}(k,p) \succ 0$, $\xi_i^{(\iota)-}(k,p) \prec 0$, $\xi_i^{-}(k,p) \prec 0$, $\rho_i(p) \prec 0$ *such that (51), (16), and*

$$\begin{array}{c} v_i(k,p) \succeq \epsilon e_n, \\ \gamma_i(k) \leq \delta\epsilon, \\ \delta \sum_{\iota=1}^{m_i} \xi_i^{(\iota)-}(k,p) e_{m_i}^{(\iota)T} e_{m_p} - \delta \sum_{\iota=1}^{m_i} \xi_i^{(\iota)+}(k,p) e_{m_i}^{(\iota)T} e_{m_p} - \eta e_{m_i}^T \rho_i(p) e_n \succeq 0, \end{array} \tag{54}$$

hold $\forall p \in \mathbb{S}, \forall (i,j) \in \{1,\ldots,N\} \times \{1,\ldots,N\}, i \neq j$, *then the constraints in (8) are handled under the control law (52).*

Replacing the term $\underline{B}_i^T(p) \sum_{q=1}^{S} \pi_{pq} v_i(k,q) + \varrho_i^-(p)$ *in Theorem 6 by* $\rho_i(p)$, *the proof of Theorem 8 can be given using a similar method to that in* Theorem 6.

Remark 7. In [41], the DMPC of MJSs was investigated based on quadratic Lyapunov functions and linear matrix inequalities. The DMPC of MJSs is proposed in this section using a positive system approach. Different from the DMPC in [41], a linear Lyapunov function associated with linear programming is used in Theorems 5–8. The main advantage of the DMPC in Theorems 5–8 is that a linear programming based DMPC algorithm is employed. As we all know, linear matrix inequalities based predictive algorithm will lead to a heavy computation burden. Especially, for large-scale computation, linear matrix inequalities have a low capacity. Linear programming overcomes the drawbacks of linear matrix inequalities. On one hand, the linear programming based conditions have a simple form. On the other hand, linear programming is powerful for dealing with large-scale computation. These points improve the traditional DMPC approach.

Remark 8. The obtained results in Theorems 5–8 imply that the sign of the states will be kept invariant under the designed DMPC controller. That is to say, the corresponding state component is nonnegative (non-positive) if some component of initial conditions is nonnegative (non-positive). Thus, the states are easy to be caught by the initial condition. It should also be pointed out that the extensions in this section have a prerequisite that the considered systems can be positively stabilized. However, not all systems satisfy the prerequisite. Up to now, there is no method to judge which class of systems can be positively stabilized. This brings conservatism to the obtained results. In practice, one may decide whether the results are available by checking the validity of the conditions in theorems.

This paper has proposed the DMPC design for PMJSs and then developed the obtained approach for MJSs. All considered systems are linear. As we all know, nonlinear systems have advantages in modeling practical dynamic processes [44,45] with respect to linear systems. There have also been some results on nonlinear PMJSs [20,21], the nonlinear MPC [28,46], and the MPC of nonlinear MJSs [27]. For positive systems, few efforts are devoted to the topics mentioned above. There are three interesting issues in future work: (i) how to developed the DMPC framework proposed in this paper for nonlinear PMJSs and (ii) how to construct a nonlinear MPC (DMPC) framework for nonlinear PMJSs.

5. Illustrative example

Susceptible-exposed-infected-removed (SEIR) is one of the most commonly used models in epidemics [47]. In such a model, there are four classes of individuals: (i) Susceptible individuals (named S) who have a large possibility to contract the disease; (ii) Asymptomatic but infectious individuals, also called exposed individuals (named E) who may transmit the disease to S; (iii) Symptomatic but infectious individuals (named I) who may transmit the disease to S; and (iii) Recovered individuals (named R) who are permanently immune, the recovery, or death. Such a simple model represents well a generic behavior of epidemics, and a related advantage consists in a small number of parameters to identify. In [48], a modified SEIR discrete-time model was proposed to model the epidemics trend of COVID-19 in China:

$$S(k+1) = S(k) - b^{\frac{p_c I(k)+r(k)E(k)}{N}} S(k), \tag{55a}$$

$$E(k+1) = (1-\sigma)E(k) + b^{\frac{p_c I(k)+r(k)E(k)}{N}} S(k), \tag{55b}$$

$$I(k+1) = (1-\chi)I(k) + \sigma E(k), \tag{55c}$$

$$R(k+1) = R(k) + \chi I(k), \tag{55d}$$

where $k \in \mathbb{N}$ is the time counted in days, N denotes the total population, the parameter $0 < \chi < +\infty$ represents the mortality and recovery rates, the parameter $0 < b < +\infty$ corresponds to the infection rate of the virus transmission from infectious to susceptibilities, $0 < \sigma < +\infty$ is the incubation rate by which the exposed develop symptoms, $0 < p_c < +\infty$ corresponds to the number of contacts for the infectious I, and $p_c \le r(k) < +\infty$ is the number of contacts per person per day for the exposed population E.

It is not hard to know that the system (55) is positive since the number of four classes individuals is nonnegative. For the corresponding analysis and synthesis, it is reasonable to employ a positive system approach. In [9], a positive system approach has been used to model HIV mitigating virus escape process. It is worthy noting that HIV is essentially a epidemic. This further reveals that positive systems play a key role in modeling epidemics. Note the fact that $R(k)$ is easy to be obtained when $I(k)$ is known. Therefore, the SEIR model (55) is modified as SEI model (55a)–(55c). It is also necessary to point out that the exposed and infectious population will affect the susceptible population, the symptomatic persons will affect the asymptomatic persons, and the susceptible persons may become the symptomatic persons. In addition, the inequality $0 < b^{\frac{p_c I(k)+r(k)E(k)}{N}} < 1$ holds. Based on these points, the SEIR model (55) is rewritten as:

$$\begin{aligned} S(k+1) &= a_{11}S(k) + a_{12}E(k) + a_{13}I(k), \\ E(k+1) &= a_{21}S(k) + a_{22}E(k) + a_{23}I(k), \\ I(k+1) &= a_{31}S(k) + a_{32}E(k) + a_{33}I(k), \end{aligned} \tag{56}$$

where $a_{11} = 1 - b^{\frac{p_c I(k)+r(k)E(k)}{N}}$, $a_{22} = 1-\sigma$, $a_{32} = \sigma$, $a_{33} = 1-\chi$ and a_{12}, a_{13}, a_{21}, a_{23}, a_{31} are unknown nonnegative weighted coefficients. The system (56) is a predicted model to estimate the population of four classes of individuals. Indeed, a more important issue is how to contain the deterioration of epidemic. Therefore, it is necessary to introduce some effective control strategies for (56). Quarantine is one of available strategies in the absence of

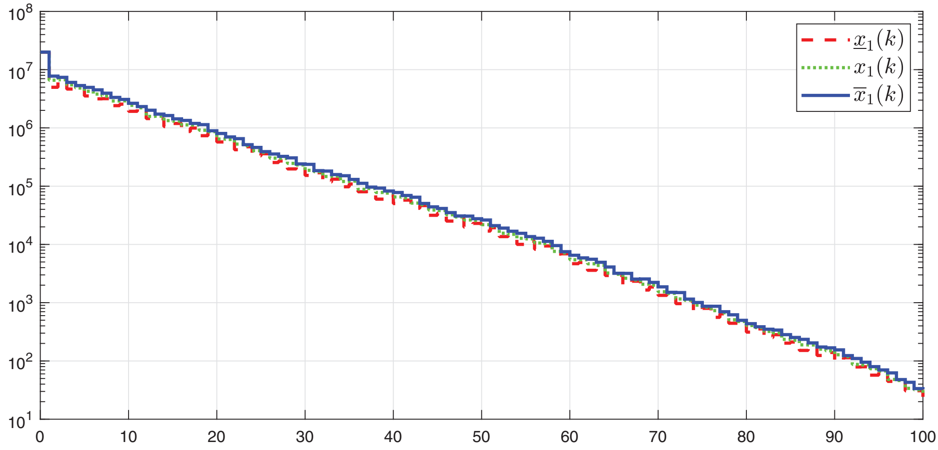

Fig. 1. The simulations of the state $x_1(k)$ and its upper and lower bounds.

specific drugs and vaccines. From the viewpoint of control theory, quarantine is to move the people from the infection zone to a safe zone or restrict their behaviors, that is, the control input $u(k) \prec 0$. In different zones, several SEI models can be established. Meanwhile, different individuals in different zones will correlate with each other. A Markov jump process is more suitable for modeling the dynamics of epidemics. In existing literature, such as [47] and [48], some identification methods were used to obtain the values of parameters b, p_c, $r(k)$, σ, χ. Considering that the measured data and parameters contain numerous uncertainties, it is difficult to make a reasonable prediction based on the SEIR model with fixed values of parameters. Thus, an interval approach has already been applied to SEIR models in [49] and [50].

By these analysis above, the system (1) with interval uncertainty is employed to re-construct SEIR for epidemics, where

$$\underline{A}(1) = \begin{pmatrix} 0.34 & 0.36 & 0.35 \\ 0.35 & 0.33 & 0.36 \\ 0.32 & 0.35 & 0.34 \end{pmatrix}, \overline{A}(1) = \begin{pmatrix} 0.45 & 0.37 & 0.36 \\ 0.36 & 0.44 & 0.37 \\ 0.42 & 0.36 & 0.35 \end{pmatrix}, \underline{B}(1) = \begin{pmatrix} 0.01 & 0.01 \\ 0.02 & 0.02 \\ 0.02 & 0.02 \end{pmatrix},$$

$$\overline{B}(1) = \begin{pmatrix} 0.05 & 0.05 \\ 0.04 & 0.04 \\ 0.03 & 0.03 \end{pmatrix},$$

and

$$\underline{A}(2) = \begin{pmatrix} 0.35 & 0.34 & 0.36 \\ 0.36 & 0.33 & 0.35 \\ 0.35 & 0.32 & 0.34 \end{pmatrix}, \overline{A}(2) = \begin{pmatrix} 0.36 & 0.35 & 0.37 \\ 0.37 & 0.34 & 0.36 \\ 0.36 & 0.33 & 0.35 \end{pmatrix}, \underline{B}(2) = \begin{pmatrix} 0.02 & 0.02 \\ 0.01 & 0.01 \\ 0.03 & 0.03 \end{pmatrix},$$

$$\overline{B}(2) = \begin{pmatrix} 0.03 & 0.03 \\ 0.03 & 0.03 \\ 0.06 & 0.06 \end{pmatrix}.$$

Give the initial condition $x(k) = (2 \times 10^6 \;\; 5 \times 10^5 \;\; 4 \times 10^4)^T$. Using Algorithm 1 gives $\hbar = 2.11$. Then, implement Algorithm 2 via 40 iterations. Here, the variables in the first predicted

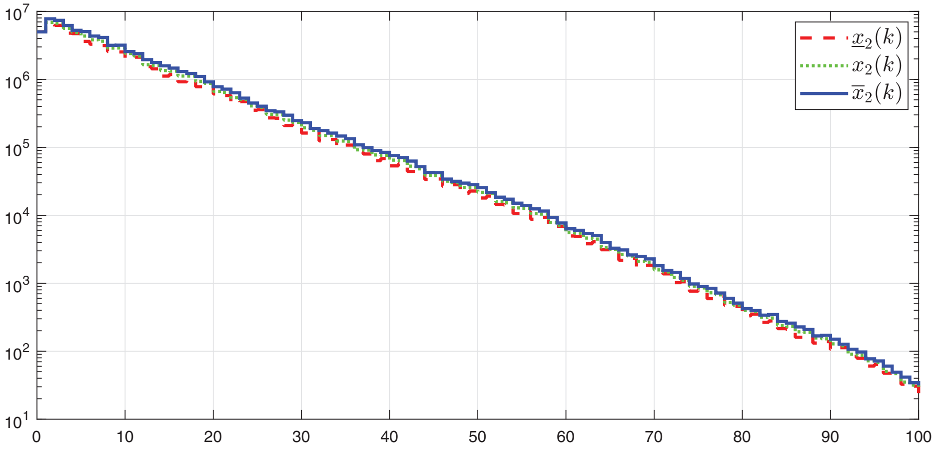

Fig. 2. The simulations of the state $x_2(k)$ and its upper and lower bounds.

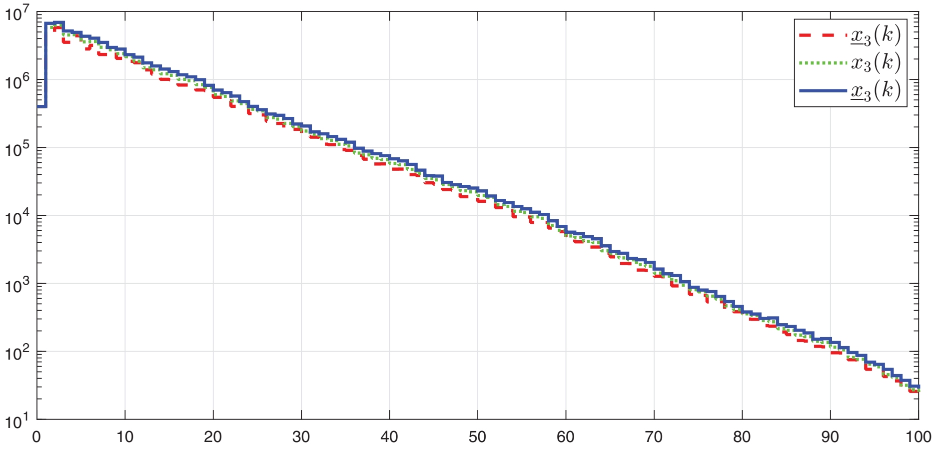

Fig. 3. The simulations of the state $x_3(k)$ and its upper and lower bounds.

step are obtained:

$$v_1(0,1) = \begin{pmatrix} 0.1498 \\ 0.1400 \\ 0.1330 \end{pmatrix}, \; v_1(0,2) = \begin{pmatrix} 0.1497 \\ 0.1399 \\ 0.1329 \end{pmatrix}, \; v_2(0,1) = \begin{pmatrix} 0.1369 \\ 0.1359 \\ 0.1327 \end{pmatrix},$$

$$v_2(0,2) = \begin{pmatrix} 0.1368 \\ 0.1358 \\ 0.1326 \end{pmatrix},$$

$$\xi_1^{(1)}(0,1) = \begin{pmatrix} -0.0124 \\ -0.0205 \\ -0.0222 \end{pmatrix}, \; \xi_1^{(1)}(0,2) = \begin{pmatrix} -0.0123 \\ -0.0204 \\ -0.0222 \end{pmatrix}, \; \xi_2^{(1)}(0,1) = \begin{pmatrix} -0.0131 \\ -0.0128 \\ -0.0118 \end{pmatrix},$$

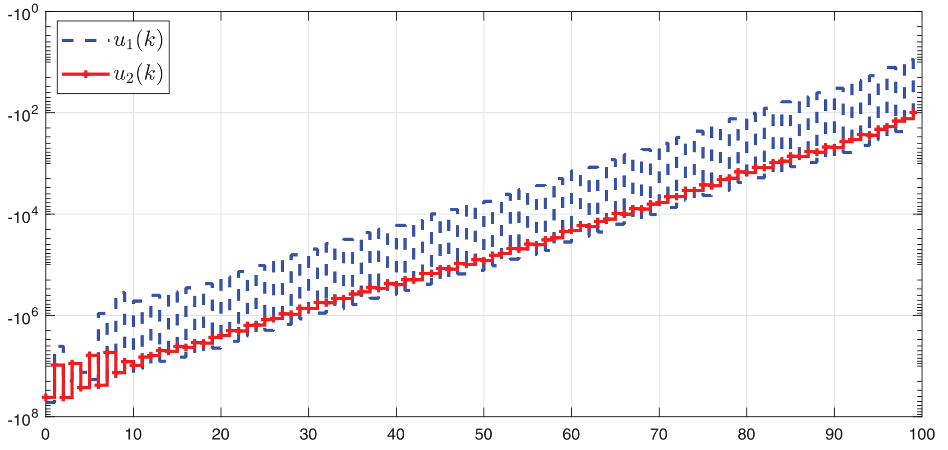

Fig. 4. The simulations of the control input $u(k)$.

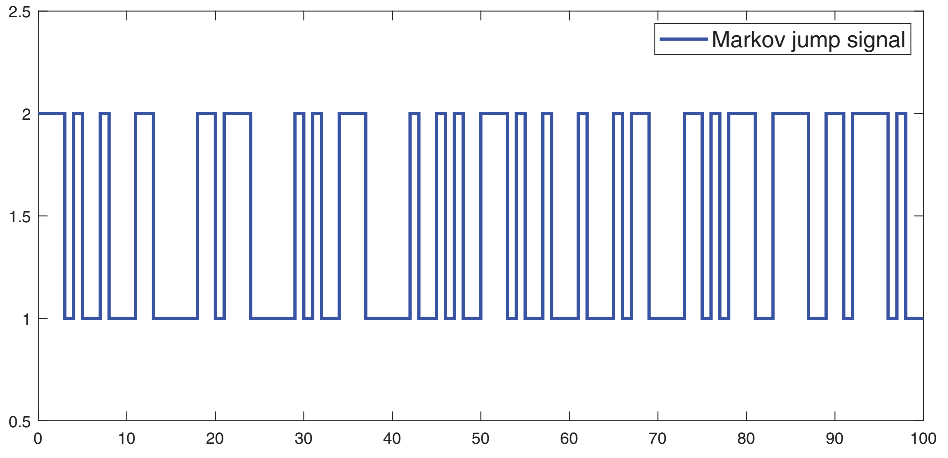

Fig. 5. One of Markov jump signals.

$$\xi_2^{(1)}(0, 2) = \begin{pmatrix} -0.0130 \\ -0.0127 \\ -0.0117 \end{pmatrix}.$$

Then, the DMPC controller gain matrices are

$$F_1^T(0, 1) = \begin{pmatrix} -1.8056 \\ -2.9896 \\ -3.2535 \end{pmatrix}, \ F_1^T(0, 2) = \begin{pmatrix} -0.0130 \\ -0.0127 \\ -0.0117 \end{pmatrix}, F_2^T(0, 1) = \begin{pmatrix} -1.6381 \\ -1.6036 \\ -1.4810 \end{pmatrix},$$

$$F_2^T(0, 2) = \begin{pmatrix} -0.1982 \\ -0.1982 \\ -1.1749 \end{pmatrix}.$$

Denote $\overline{x}(k) = (\overline{x}_1(k) \ \overline{x}_2(k) \ \overline{x}_3(k))^T$ and $\underline{x}(k) = (\underline{x}_1(k) \ \underline{x}_2(k) \ \underline{x}_3(k))^T$ as the upper and lower bounds of the state $x(k)$, respectively. Figs. 1–3 show the simulations of the states and their lower and upper bounds, Fig. 4 is the control input, and Fig. 5 is the Markov jump signal. From Figs. 1–3, it is clear that the infectious individuals are contained in a limited scope. From Fig. 4, the quarantine population is much at the first 30 sample time instants (days) and it is few after 60 days. In Fig. 4, the quarantine population at the beginning of spread of epidemics is more than the sum of the susceptible, asymptomatic, and symptomatic infectious individuals. In practice, this is unreasonable. Indeed, it means that all individuals are required to implement the quarantine strategy if the quarantine population in Fig. 4 is more than the sum of the susceptible, asymptomatic, and symptomatic infectious individuals.

Remark 9. The literature [47–50] was concerned with the modeling, the parameter identification of models, and the state estimation. These literature can provide some available modeling of SEIR and present some effective predict for the trend of epidemics. However, few strategies are devoted to how to contain epidemics. It is fundamentally important to propose effective approaches to suppress the spread of epidemics. In this section, a suggestive DMPC approach is given to fill the mentioned gap. It should be pointed out that the parameters in the considered system are not from a real case in some zone. In practice, one can utilize the methods in [47–50] to identify parameters by virtue of some real data. Then, the approach in this section can be used to contain epidemics.

6. Conclusions and future work

This paper has presented a DMPC framework for PMJSs. Different from the DMPC of MJSs, the elements of the DMPC framework of PMJSs are all linear. Using matrix decomposition techniques, the DMPC controller is designed in terms of linear programming to guarantee the positivity and stochastic stability of the systems. The interval and polytopic uncertainties are handled, respectively. Some corresponding algorithms are provided to check the presented conditions.

The proposed DMPC framework can be further developed for the corresponding issues of positive systems such as positive Takagi-Sugeno fuzzy systems, positive multi-agent systems, and so on. It is also interesting to establish a DMPC framework on positive Markovian systems with disturbances. In this paper, the DMPC control law of PMJSs is required to be negative. How to remove the sign restriction of the DMPC control law may be a significant topic in future work.

Declaration of Competing Interest

The authors declare that they have no competing interest concerning the publication of this paper.

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